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Liouville field theory with heavy charges. I. The pseudosphere*

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Abstract

We work out the perturbative expansion of quantum Liouville theory on the pseudosphere starting from the semiclassical limit of a background generated by heavy charges. By solving perturbatively the Riemann-Hilbert problem for the Poincaré accessory parameters, we give in closed form the exact Green function on the background generated by one finite charge. Such a Green function is used to compute the quantum determinants i.e. the one loop corrections to known semi-classical limits thus providing the resummation of infinite classes of standard perturbative graphs. The results obtained for the one point function are compared with the bootstrap formula while those for the two point function are compared with the existing double perturbative expansion and with a degenerate case, finding complete agreement.

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Introduction

The conformal bootstrap program has provided very deep results in conformal field theory and in particular in Liouville quantum field theory [1, 2, 3]. In this approach one assumes at the outset conformal invariance and, by using formal properties of the functional integral and some other assumptions, one arrives at functional equations for the correlation functions. Under reasonable regularity assumptions their solution provide the exact correlation functions.

Here and in an accompanying paper we address the problem to recover the conformal theory from the usual field theoretic procedure in which one starts from a stable background and then one integrates over the quantum fluctuations. As it is well known, a quantum field theory is specified not only by an action but also by a regularization and renormalization procedure. In [4, 5, 6] it was found that not all the regularization procedures provide a theory which is invariant under the full conformal group. The regularization suggested at the perturbative level in [1] in the case of the pseudosphere provides the vertex functions with the correct quantum dimensions [7] at the first perturbative order. Here it is explicitly proved that such a result stays unchanged to all orders perturbation theory. In particular the weight of the cosmological term becomes $(1, 1)$ as required by the invariance under local conformal transformations.

The pseudosphere case was already considered in [1] and more fully developed in [4, 8]. These calculations correspond to a double perturbative expansion in the coupling constant and in the charge of the vertex function.

Here instead we start from the background generated by finite charges, i.e. "heavy charges" in the terminology of [3]. This means that we consider the vertex operators $V_{\alpha_n}(z_n) = e^{2\alpha_n \phi(z_n)}$ with $\alpha_n = \eta_n/b$ and η_n fixed in the semiclassical limit $b \to 0$.

In the case of a single heavy charge, by solving a Riemann-Hilbert problem we are able to compute the exact Green function on such a background in closed form, and the Green function is used to develop the subsequent perturbative expansion in the coupling constant for the one and two point functions. In this way we obtain a resummation of infinite classes of perturbative graphs.

Some of the results derived here were reported in [9]. In the present paper we give full details of the computational procedure. In section 1 we lay down the notations and discuss the semiclassical limit. In section 2 it is shown that the regularization procedure of [1] provides the vertex functions with the correct quantum dimensions to all orders perturbation theory. In section 3 we solve the Riemann Hilbert problem which allows the determination of the exact Green function on the background given by one heavy

charge. In section 4 the one loop correction to the semiclassical one point function is computed. The result is compared with the expansion of the exact one point function derived in the bootstrap approach [1] finding complete agreement. In section 5 the two point function with one finite charge and an infinitesimal one is computed by employing analogous technique. Particular cases of such new result are compared with the existing double perturbative expansion in α and b and with a degenerate case, finding complete agreement in both cases. In appendix A we derive the behavior of the conformal factor for the N point classical background at infinity and in appendix B we give the details of the computation of the Green function.

1 Classical and quantum action on the pseudosphere

Let us consider the geometry of the pseudosphere in the representation of the unit disk $\Delta = \{z \in \mathbb{C}\,;|z| < 1\}$.

We write the N point function for the vertex operators $V_\alpha(z) = e^{2\alpha\phi(z)}$ in the form

$$\left\langle\, e^{2\alpha_1\phi(z_1)} \dots e^{2\alpha_N\phi(z_N)}\,\right\rangle \;=\; \frac{1}{Z}\int \mathcal{D}\,[\,\phi\,]\;\, e^{-S_{\Delta,N}[\,\phi\,]} \tag{1.1}$$

where

$$Z \;=\; \int \mathcal{D}\,[\,\phi\,]\;\, e^{-S_{\Delta,N=0}[\,\phi\,]} \tag{1.2}$$

and $S_{\Delta,\,N}[\,\phi\,]$ is the action of Liouville field theory on the pseudosphere in presence of N sources, which is given by the following expression [3, 4]

$$\begin{aligned} S_{\Delta,\,N}[\,\phi\,] \;\;=\;\; \lim_{\substack{\varepsilon_n \to 0 \\ r \to 1}} \; & \left\{ \int_{\Delta_{\,r,\,\varepsilon}} \left[\frac{1}{\pi}\,\partial_z\phi\,\partial_{\bar z}\phi + \mu\, e^{2b\phi}\right] d^2 z \right. \\ & - \frac{Q}{2\pi i}\oint_{\partial\Delta_r} \phi\left(\frac{\bar z}{1 - z\bar z}\,dz - \frac{z}{1 - z\bar z}\,d\bar z\right) + f(r,b) \\ & \left. - \frac{1}{2\pi i}\sum_{n=1}^{N}\alpha_n\oint_{\partial\gamma_n}\phi\left(\frac{dz}{z - z_n} - \frac{d\bar z}{\bar z - \bar z_n}\right) - \sum_{n=1}^{N}\alpha_n^2\log\varepsilon_n^2 \right\} \end{aligned} \tag{1.3}$$

with $Q = 1/b + b$ and $d^2 z = i dz \wedge d\bar z/2$.

The integration domain $\Delta_{\,r,\,\varepsilon} \;=\; \Delta_{\,r} \setminus \bigcup_{n=1}^N \gamma_n$ is obtained by removing N disks $\gamma_n \;=\; \{|z - z_n| < \varepsilon_n\}$ from the disk $\Delta_{\,r} = \{\,|z| < r < 1\,\} \subset \Delta$. The boundary behaviors of the field ϕ are given by

$$\begin{aligned} \phi(z) \;&=\; -\frac{Q}{2}\log\,(\,1 - z\bar z\,)^2 + O(1) \qquad \text{when} \qquad |z| \to 1 && (1.4) \\ \phi(z) \;&=\; -\,\alpha_n\,\log\,|\,z - z_n\,|^2\, + O(1) \qquad \text{when} \qquad z \,\to z_n\,. && (1.5) \end{aligned}$$

The function $f(r,b)$ is a subtraction term independent of the field ϕ and of the charges.

In order to connect the quantum theory to its semiclassical limit it is useful to define [3]

$$\varphi \;=\; 2b\phi\;, \qquad\qquad \alpha_n \;=\; \frac{\eta_n}{b}\;. \tag{1.6}$$

The charges $\alpha_n = \eta_n/b$ are called heavy charges [3] because in the perturbative limit $b \to 0$ the parameters η_n are kept fixed and therefore α_n diverge. Since the measure is $e^{\varphi} d^2z$, the condition of local finiteness of the area around each source and the asymptotic behavior (1.5) for the field ϕ impose that $1-2\eta_n > 0$ [10, 11].

Now we decompose the field φ as the sum of a classical background field φ_B and a quantum field

$$\varphi \;=\; \varphi_B + 2b\,\chi\;. \tag{1.7}$$

Then, we can write the action as the sum of a classical and a quantum action as follows

$$S_{\Delta,\,N}[\,\phi\,] \;=\; S_{cl}[\,\varphi_B\,] + S_q[\,\varphi_B,\,\chi\,]\;. \tag{1.8}$$

The classical action is given by

$$\begin{aligned} S_{cl}[\,\varphi_B\,] \;&=\; \frac{1}{b^2}\lim_{\substack{\varepsilon_n\to 0\\ r\to 1}} \left\{ \int_{\Delta_{r,\varepsilon}} \left[\frac{1}{4\pi}\,\partial_z\varphi_B\,\partial_{\bar z}\varphi_B + \mu b^2 e^{\varphi_B}\right] d^2z \right. \\ &\quad - \frac{1}{4\pi i}\oint_{\partial\Delta_r}\varphi_B\left(\frac{\bar z}{1-z\bar z}\,dz - \frac{z}{1-z\bar z}\,d\bar z\right) + f_{cl}(r,\mu b^2) \\ &\quad \left. - \frac{1}{4\pi i}\sum_{n=1}^{N}\eta_n\oint_{\partial\gamma_n}\varphi_B\left(\frac{dz}{z-z_n} - \frac{d\bar z}{\bar z-\bar z_n}\right) - \sum_{n=1}^{N}\eta_n^2\log\varepsilon_n^2 \right\} \end{aligned} \tag{1.9}$$

while the quantum action reads

$$\begin{aligned} S_q[\,\varphi_B\,,\chi\,] \;&=\; \lim_{\substack{\varepsilon_n\to 0\\ r\to 1}} \left\{ \int_{\Delta_{r,\varepsilon}} \left[\frac{1}{\pi}\,\partial_z\chi\,\partial_{\bar z}\chi + \mu\, e^{\varphi_B}\left(e^{2b\chi}-1\right) - \frac{1}{\pi b}\,\chi\,\partial_z\partial_{\bar z}\varphi_B\right] d^2z \right. \\ &\quad - \frac{1}{2\pi i\, b}\oint_{\partial\Delta_r}\chi\left(\frac{\bar z}{1-z\bar z} - \frac{1}{2}\,\partial_z\varphi_B\right)dz + \frac{1}{2\pi i\, b}\oint_{\partial\Delta_r}\chi\left(\frac{z}{1-z\bar z} - \frac{1}{2}\,\partial_{\bar z}\varphi_B\right)d\bar z \\ &\quad - \frac{1}{4\pi i}\oint_{\partial\Delta_r}\varphi_B\left(\frac{\bar z}{1-z\bar z}\,dz - \frac{z}{1-z\bar z}\,d\bar z\right) + f_q(r,\mu b^2) \\ &\quad - \frac{b}{2\pi i}\oint_{\partial\Delta_r}\chi\left(\frac{\bar z}{1-z\bar z}\,dz - \frac{z}{1-z\bar z}\,d\bar z\right) \\ &\quad \left. - \frac{1}{2\pi i\, b}\sum_{n=1}^{N}\oint_{\partial\gamma_n}\chi\left(\frac{\eta_n}{z-z_n} + \frac{1}{2}\,\partial_z\varphi_B\right)dz + \frac{1}{2\pi i\, b}\sum_{n=1}^{N}\oint_{\partial\gamma_n}\chi\left(\frac{\eta_n}{\bar z-\bar z_n} + \frac{1}{2}\,\partial_{\bar z}\varphi_B\right)d\bar z \right\}. \end{aligned} \tag{1.10}$$

We remark that the subtraction terms $f_{cl}(r,\mu b^2)$ and $f_q(r,\mu b^2)$ are independent of the fields and of the charges η_n.

For the classical background field near the sources we have

$$\varphi_B(z) \;=\; -\,2\eta_n \log|z-z_n|^2 \;+O(1) \qquad \text{when} \quad z\,\longrightarrow z_n \tag{1.11}$$

while in appendix A the following boundary behavior for $\varphi_B(z)$ is proved

$$\varphi_B(z) \;=\; -\,\log\,(1-z\bar z)^2 + f(\mu b^2) + O\big((1-z\bar z)^2\big) \qquad \text{when} \quad |z|\longrightarrow 1 \tag{1.12}$$

where $f(\mu b^2)$ is a constant depending on μb^2.

Comparing (1.11) with (1.5), we see that χ is regular at the sources. This fact and the boundary behavior (1.11) imply the vanishing of the last line in (1.10) in the limit $\varepsilon_n \longrightarrow 0$. Moreover, since the field χ can diverge only like a logarithm when $z\bar z\longrightarrow 1$, the asymptotic (1.12) implies that the second line in (1.10) vanishes in the limit $r\longrightarrow 1$.

Now we focus on the classical action $S_{cl}[\,\varphi_B\,]$. The vanishing of its first variation with respect to the field φ_B with boundary conditions (1.12) and (1.11) gives the Liouville equation in presence of N sources

$$-\,\partial_z\partial_{\bar z}\,\varphi_B \;+2\pi\,\mu b^2\,e^{\,\varphi_B} \;=\; 2\pi\sum_{n=1}^{N}\eta_n\,\delta^2(z-z_n)\;. \tag{1.13}$$

Under a generic conformal transformation $z\,\longrightarrow\, w(z)$ the background field changes as follows

$$\varphi_B(z) \qquad\longrightarrow\qquad \tilde\varphi_B(w) \;=\; \varphi_B(z)\;-\;\log\left|\frac{dw}{dz}\right|^2 \tag{1.14}$$

so that $e^{\,\varphi_B}d^2z$ is invariant.

In particular, under a $SU(1,1)$ transformation, which maps the unit disk Δ into itself, the classical action (1.9) changes as follows [4, 12]

$$\widetilde S_{cl}[\,\tilde\varphi_B\,] \;=\; S_{cl}[\,\varphi_B\,] + \sum_{n\,=\,1}^{N}\frac{\eta_n(\,1-\eta_n)}{b^2}\;\log\left|\frac{dw}{dz}\right|^2_{z\,=\,z_n}\;. \tag{1.15}$$

The classical action $S_{cl}[\,\varphi_B\,]$ computed on the solution φ_B of the equation of motion (1.13) becomes a function $S_{cl}(\eta_1,z_1;\ldots;\eta_N,z_N)$ of the positions z_n of the sources and of their charges η_n. This function provides the semiclassical expression of the N point function for the Liouville vertex operators $V_\alpha(z)=e^{2\alpha\phi(z)}$

$$\langle\, V_{\alpha_1}(z_1)\ldots V_{\alpha_N}(z_N)\,\rangle_{sc} \;=\; \frac{e^{-S_{cl}(\eta_1,\,z_1;\,\ldots\,;\,\eta_N,\,z_N)}}{e^{-S_{cl}(0)}}\;. \tag{1.16}$$

The function in the denominator is the semiclassical contribution of the partition function Z in (1.1) and it is $SU(1,1)$ invariant.

By using (1.15), we immediately see that (1.16) has the following transformation properties under $SU(1,1)$

$$\langle\, \widetilde{V}_{\alpha_1}(w_1)\ldots \widetilde{V}_{\alpha_N}(w_N)\,\rangle_{sc} \;=\; \prod_{n=1}^{N} \left|\frac{dw}{dz}\right|_{z=z_n}^{-\,2\,\eta_n(1-\eta_n)/b^2} \langle\, V_{\alpha_1}(z_1)\ldots V_{\alpha_N}(z_N)\,\rangle_{sc}\,. \tag{1.17}$$

This means that the semiclassical dimensions of the vertex operator $V_\alpha(z)$ are $\eta(1-\eta)/b^2 = \alpha\,(1/b-\alpha)$.

Now we consider the quantum action (1.10). For a background field φ_B satisfying the Liouville equation with sources (1.13) and the boundary conditions (1.12) and (1.11), the quantum action (1.10) becomes

$$\begin{aligned} S_q[\,\varphi_B\,,\chi\,] \;&=\; \lim_{r\to 1}\left\{ \int_{\Delta_r}\left[\frac{1}{\pi}\,\partial_z\chi\,\partial_{\bar z}\chi + \mu\,e^{\varphi_B}\left(e^{2b\,\chi} - 1 - 2b\,\chi\right)\right] d^2z \right. \\ &\qquad \left. -\,\frac{b}{2\pi i}\oint_{\partial\Delta_r}\chi\left(\frac{\bar z}{1-z\bar z}\,dz \,-\, \frac{z}{1-z\bar z}\,d\bar z\right)\right\}. \end{aligned} \tag{1.18}$$

If the Green function vanishes quadratically on the boundary, the last term in (1.18) does not contribute to the perturbative expansion. In section 3 we shall verify this fact explicitly for the case $N = 1$.

In this paper, we are mainly interested in the case of a single source, i.e. $N = 1$ and $\eta_1 = \eta$. By exploiting the invariance under $SU(1,1)$ we can place the source in $z_1 = 0$. In this case the background field, i.e. the solution of the Liouville equation (1.13) with boundary behaviors (1.12) and (1.11), can be explicitly written [10, 13]

$$e^{\varphi_{cl}} \;=\; \frac{1}{\pi\mu b^2}\,\frac{(1-2\eta)^2}{\left((z\bar z)^{\eta} - (z\bar z)^{1-\eta}\right)^2}\;. \tag{1.19}$$

It is important to notice that the behavior of φ_{cl} on the boundary $\partial\Delta$, i.e. at infinity, is independent of η both in the divergent term and in the constant term

$$\varphi_{cl} \;=\; -\,\log(1-z\bar z)^2 -\,\log\left(\pi\mu b^2\right) \,+\, O\big((1-z\bar z)^2\big) \qquad \text{when} \qquad z\bar z \to 1\,. \tag{1.20}$$

Notice that the term $O(1 - z\bar z)$ is also absent, in agreement with the asymptotics (1.12) for the background field.

In appendix A we prove that this is a general feature for the solution φ_B in presence of N sources; therefore the two boundary integrals in the second line of (1.10) vanish when $|z| \to 1$, being χ logarithmically divergent at most.

By using the explicit form of the classical background field (1.19), we can write the

expression of the semiclassical one point function for a vertex operator with charge $\alpha = \eta/b$ placed in 0

$$\begin{aligned}\left\langle V_{\eta/b}(0) \right\rangle_{sc} &= \exp\left\{ - S_{cl}[\varphi_{cl}] + S_{cl}[\varphi_{cl}]|_{\eta=0} \right\} \qquad (1.21)\\ &= \exp\left\{ -\frac{1}{b^2}\left(\eta \log\left(\pi\mu b^2\right) + 2\eta + (1-2\eta)\log(1-2\eta) \right) \right\} .\end{aligned}$$

The one point function in the basic vacuum found by Zamoldchikov and Zamolodchikov [1] within the conformal bootstrap approach is

$$\langle V_\alpha(z_1) \rangle = \frac{U_{1,1}(\alpha)}{(1 - z_1 \bar{z}_1)^{2\alpha(Q-\alpha)}} \qquad (1.22)$$

with

$$U_{1,1}(\alpha) = \left(\pi\mu\gamma(b^2)\right)^{-\alpha/b} \frac{\Gamma(Q\,b)\,\Gamma(Q/b)\,Q}{\Gamma\big((Q-2\alpha)\,b\big)\,\Gamma\big((Q-2\alpha)/b\big)\,(Q-2\alpha)} \qquad (1.23)$$

where $Q = 1/b + b$ and $\gamma(x) = \Gamma(x)/\Gamma(1-x)$.
The expression (1.21) agrees with the semiclassical term of (1.22) for $z_1 = 0$ and $\alpha = \eta/b$.

2 The quantum dimensions

In this section we show that the quantum determinant of the N point function provides the quantum correction to the conformal dimensions and that no further contributions to the conformal dimensions occur.
The $O(b^0)$ quantum correction to the N point function $\langle V_{\alpha_1}(z_1) \dots V_{\alpha_N}(z_N) \rangle$ is given by the quantum determinant

$$\big(\operatorname{Det} D\big)^{-1/2} \equiv \frac{1}{Z_0} \int \mathcal{D}[\chi] \; \exp\left\{ -\frac{1}{2} \int_\Delta \chi \left(-\frac{2}{\pi}\,\partial_z\partial_{\bar z} + 4\mu b^2\, e^{\varphi_B} \right) \chi \, d^2 z \right\} \qquad (2.1)$$

where φ_B is the classical background field solving the Liouville equation and with asymptotics (1.12) and (1.11), while Z_0 is the quadratic part of the partition function (1.2).
Taking the logarithmic derivative w.r.t. η_j of $\big(\operatorname{Det} D\big)^{-1/2}$, we find the following integral

$$\frac{\partial}{\partial \eta_j} \log\big(\operatorname{Det} D\big)^{-1/2} = -2 \int_\Delta g(z,z)\, \frac{\partial\big(\mu b^2 e^{\varphi_B}\big)}{\partial \eta_j}\, d^2 z \qquad j = 1, \dots, N \qquad (2.2)$$

where $g(z,t)$ is the Green function on the classical background described by φ_B and, due to the boundary behavior (1.12), we have that

$$\frac{\partial\big(\mu b^2 e^{\varphi_B}\big)}{\partial \eta_j} = O(1) \qquad \text{when} \quad |z| \to 1 . \qquad (2.3)$$

This asymptotic behavior can be explicitly checked for the conformal factor (1.19) of the $N = 1$ case. Formula (2.2) exposes the key role of the Green function at coincident points. Obviously $g(z,z)$ has to be regularized and we shall show in what follows that the regularization proposed by Zamolodchikov and Zamolodchikov (ZZ) in [1], i.e.

$$g(z,z) \equiv \lim_{t \to z} \left\{ g(z,t) + \frac{1}{2} \log |\, z - t\,|^2 \right\} \tag{2.4}$$

gives rise to the correct quantum dimensions. To this end we examine the transformation properties of the quantum determinant.

From the equation satisfied by the Green function, that is

$$D\, g(z,z') \equiv \left(-\frac{2}{\pi}\, \partial_z \partial_{\bar z} \,+\, 4\, \mu b^2\, e^{\varphi_B} \right) g(z,z') \,=\, \delta^2(z - z'\,) \tag{2.5}$$

we see that, under a $SU(1,1)$ transformation

$$z \;\longrightarrow\; w \,=\, \frac{a\, z + b}{\bar b\, z + \bar a} \qquad\qquad |a|^2 - |b|^2 \,=\, 1 \tag{2.6}$$

$g(z,z'\,)$ is invariant in value, i.e.

$$g(z,z') \;\longrightarrow\; \tilde g(w,w') \,=\, g(z,z'\,)\;. \tag{2.7}$$

Instead, because of the term $\log |z-t|$ in (2.4), the function $g(z,z)$ transforms as follows under $SU(1,1)$

$$g(z,z) \;\longrightarrow\; \tilde g(w,w) \,=\, g(z,z) + \frac{1}{2}\, \log \left| \frac{dw}{dz} \right|^2 \tag{2.8}$$

where $w(z)$ is given by (2.6). Then, the transformation law for the expression (2.2) is

$$\frac{\partial}{\partial \eta_j} \log \big(\,\mathrm{Det}\, \widetilde D \,\big)^{-1/2} \;=\; -\,2 \int_\Delta \tilde g(w,w)\, \frac{\partial \big(\mu b^2 e^{\varphi_B}\big)}{\partial \eta_j}\; d^2 w \tag{2.9}$$

$$=\; \frac{\partial}{\partial \eta_j} \log \big(\, \mathrm{Det}\, D\,\big)^{-1/2} \;-\; \int_\Delta \log \left| \frac{dw}{dz} \right|^2 \frac{\partial \big(\mu b^2 e^{\varphi_B}\big)}{\partial \eta_j}\; d^2 z \tag{2.10}$$

where we have used the fact that the $SU(1,1)$ transformation (2.6) does not depend on the charges η_j. The Liouville equation (1.13) allows to write the second term of (2.10) as follows

$$-\,\frac{1}{2\pi}\, \lim_{r \to 1} \left\{ \frac{\partial}{\partial \eta_j}\, \lim_{\varepsilon_n \to 0} \int_{\Delta_{\,r,\,\varepsilon}} \log \left| \frac{dw}{dz} \right|^2 \partial_z \partial_{\bar z} \varphi_B\, d^2 z \right\} \tag{2.11}$$

where the integration domain is the same occurring in (1.3). This integral can be computed integrating by parts. Since $a/b \notin \Delta$, it becomes

$$-\,\frac{\partial}{\partial \eta_j} \sum_{k=1}^{N} \eta_k\, \log \left| \frac{dw}{dz} \right|^2_{z=z_k} \tag{2.12}$$

$$+ \lim_{r \to 1} \frac{\partial}{\partial \eta_j} \left[\frac{1}{4\pi i} \oint_{\partial \Delta_r} \varphi_B\, \partial_z \log \frac{dw}{dz}\, dz \,+\, \frac{1}{4\pi i} \oint_{\partial \Delta_r} \partial_{\bar z} \varphi_B\, \log \left| \frac{dw}{dz} \right|^2 d\bar z \right]$$

where the first line comes from the circles γ_n around the sources, as φ_B behaves like in (1.11). Then, because of the asymptotic (1.12), the second line of (2.12) vanishes in the limit $r \to 1$ and we are left only with the first line.
By integrating back, we find

$$\log \left(\operatorname{Det} \widetilde{D} \right)^{-1/2} = \log \left(\operatorname{Det} D \right)^{-1/2} - \sum_{k=1}^{N} \eta_k \log \left| \frac{dw}{dz} \right|^2_{z=z_k} + f(z_1, \ldots, z_N) \qquad (2.13)$$

where $f(z_1, \ldots, z_N)$ is a function of the positions of the sources and of the transformation parameters but, since when all the charges vanish we have

$$\left(\operatorname{Det} \widetilde{D} \right)^{-1/2} \Big|_{\eta_i = 0} = \left(\operatorname{Det} D \right)^{-1/2} \Big|_{\eta_i = 0} = 1 \qquad (2.14)$$

then $f(z_1, \ldots, z_N)$ vanishes identically and the transformation law for the quantum determinant under $SU(1,1)$ reads

$$\left(\operatorname{Det} \widetilde{D} \right)^{-1/2} = \prod_{n=1}^{N} \left| \frac{dw}{dz} \right|^{-2\eta_n}_{z=z_n} \left(\operatorname{Det} D \right)^{-1/2} . \qquad (2.15)$$

Comparing this result with (1.17), we find that the semiclassical dimensions $\eta(1-\eta)/b^2$ are modified by a quantum correction to

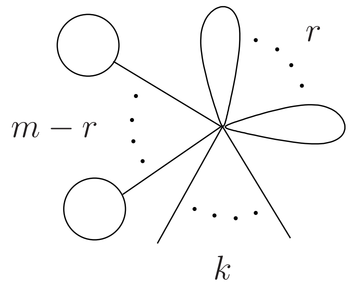
$$\Delta_\alpha = \frac{\eta(1-\eta)}{b^2} + \eta = \alpha(Q - \alpha) \qquad (2.16)$$

i.e.

$$\langle \widetilde{V}_{\alpha_1}(w_1) \ldots \widetilde{V}_{\alpha_N}(w_N) \rangle = \prod_{n=1}^{N} \left| \frac{dw}{dz} \right|^{-2\,(\eta_n(1-\eta_n)/b^2 + \eta_n)}_{z=z_n} \langle V_{\alpha_1}(z_1) \ldots V_{\alpha_N}(z_N) \rangle . \qquad (2.17)$$

The quantum conformal dimensions (2.16) are the ones found in [7] within the hamiltonian approach.

There are no further corrections to the quantum conformal dimensions (2.16) of the vertex operator $V_\alpha(z) = e^{2\alpha\phi(z)}$. This statement can be proved to all order in b by direct inspection of the perturbative graphs occurring in the expansion in α and b, along the line of [1, 4, 8]; therefore now the propagator is given by (B.22).
Since we adopt a $SU(1,1)$ non invariant regularization [1, 4], the graphs that can modify the conformal dimensions are only the ones containing tadpoles or simple loops. Let us consider a vertex which bears r simple loops, $m - r$ tadpoles and k ordinary propagators, as shown in the following figure



The order of this vertex is $k+2r+(m-r)=k+r+m$. It is generated by the interaction term

$$\int_\Delta \frac{\big(2b\chi(z)\big)^{k+m+r}}{(k+m+r)!}\; d\nu(z)\; \frac{1}{(m-r)!}\prod_{s=1}^{m-r}\left(-\int_\Delta \frac{\big(2b\chi(z_s)\big)^3}{3!}\; d\nu(z_s)\right) \tag{2.18}$$

where the measure is defined as

$$d\nu(z)\;\equiv\;\mu\, e^{\varphi_{cl}(z)|_{\eta=0}}\, d^2z\;=\;\frac{d^2z}{\pi b^2(1-z\bar{z})^2}\;. \tag{2.19}$$

The effective vertex due to (2.18) is

$$\begin{aligned}\int_\Delta d\nu(z)\,\frac{(2b)^{k+m+r}}{(k+m+r)!}\binom{k+m+r}{k}\chi^k(z)\;\times \\ \times\binom{m+r}{m-r}\left(-\frac{2^3b^3}{3!}\,3\,P(z)\right)^{m-r}(2r-1)!!\;\hat{g}(z,z)^r\end{aligned} \tag{2.20}$$

where $P(z)$ is the tadpole contribution

$$P(z)=\int_\Delta \hat{g}(z,z')\;\hat{g}(z',z')\,d\nu(z')\;. \tag{2.21}$$

Adopting the ZZ regularization procedure [1], the propagator at coincident points (i.e. the simple loop) is given by

$$\hat{g}(z,z)\;\equiv\;\lim_{t\to z}\left\{\hat{g}(z,t)+\frac{1}{2}\,\log|\,z-t\,|^2\right\}\;=\;\log(1-z\bar{z})-1\;. \tag{2.22}$$

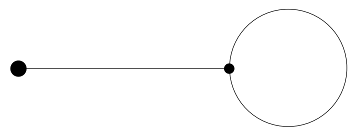
Working out the factorials in (2.20) and summing over the number r of simple loops, we have

$$\begin{aligned}\sum_{r=0}^{m}\int_\Delta d\nu(z)\;\frac{(2b\chi(z))^k}{k!}\;\frac{(2b^2)^m}{m!}\;\big(-4b^2P(z)\big)^{m-r}\binom{m}{m-r}\hat{g}(z,z)^r\;= \\ =\int d\nu(z)\;\frac{(2b\chi(z))^k}{k!}\;\frac{(2b^2)^m}{m!}\;\big(-4b^2P(z)+\hat{g}(z,z)\big)^m\;.\end{aligned} \tag{2.23}$$

Using the equation for the propagator $\hat{g}(z,z')$ is easy to show that [1]

$$-\,4\,b^2\,P(z)\,+\,\hat{g}(z,z)\;=\;\frac{1}{2}\,. \tag{2.24}$$

Notice that if one chooses a $SU(1,1)$ invariant regularization instead of (2.22), then the identity (2.24) has a vanishing right hand side.
Repeating the argument for all vertices bearing tadpoles and simple loops, we are left with a convergent graph which is invariant under $SU(1,1)$.
As shown in [4], starting from the regularized action (1.8), the exponentiation of the following graph



changes the dimensions of the vertex operator $V_\alpha(z)$ from the semiclassical value $\alpha(1/b-\alpha)$ to the value $\alpha(1/b+b-\alpha)=\alpha(Q-\alpha)$. We recall [1, 4] that in the standard approach in which one simply adds sources to the action [1], the change from the naive dimensions α/b to the semiclassical dimensions $\alpha(1/b-\alpha)$ is provided by the exponentiation of the simple loop (2.22), which is absent in the approach adopted in [4] that starts from the classical regularized action (1.8) and recovers $\alpha(1/b-\alpha)$ at the semiclassical level.

3 The Green function on the classical background

Our next aim will be to compute the exact Green function on the classical background φ_{cl} given in (1.19). We shall employ the method developed in [5]. The procedure allows also to compute the first term of the expansion in ε of the conformal factor in presence of the source in $z=0$ (Δ representation) with finite charge η and of another source with infinitesimal charge ε elsewhere.
First, we recall that the general solution of the Liouville equation in presence of N sources is given by

$$\pi\mu b^2\,e^{\varphi(z)}\;=\;\frac{|\,\omega_{12}\,|^2}{\Big(\,y_1(z)\,\overline{y_1(z)}-y_2(z)\,\overline{y_2(z)}\,\Big)^2} \tag{3.1}$$

where $y_i(z)$ are two independent solutions of the fuchsian differential equation

$$\frac{d^2y}{dz^2}\,+Q(z)\,y\;=\;0 \tag{3.2}$$

and ω_{12} is their wronskian $\omega_{12}=y_1y_2'-y_1'y_2$. The expression of $Q(z)$ is given by

$$e^{\varphi/2}\,\partial_z^2e^{-\varphi/2}\;=\;\frac{1}{4}\,(\partial_z\varphi)^2-\frac{1}{2}\,\partial_z^2\varphi\;=\;-\,Q(z)\;=\;-\,b^2\,T(z) \tag{3.3}$$

where $T(z)$ is the analytic component of the classical energy momentum tensor.
The analytic function $Q(z)$ contains both double poles, whose residues are related to the charges η_n, and simple poles, whose residues are the Poincaré accessory parameters and have to be determined by imposing the monodromy condition on the solution.
Under a change $z \to \xi(z)$ the transformation law of the solutions of (3.2) is given by

$$y(z) \quad \longrightarrow \quad \tilde{y}(\xi) \;=\; \big(z'(\xi)\big)^{-1/2}\, y\big(z(\xi)\big)\;. \tag{3.4}$$

It ensures that the wronskian and the measure $e^{\varphi(z)} dz \wedge d\bar{z}$ are separately invariant.
The energy momentum tensor must satisfy some boundary conditions guaranteeing that there is neither energy momentum flow [14, 15] nor singularity at infinity. These conditions can be formulated in a clearer way in the upper half plane $\mathbb{H} = \{\,\xi \in \mathbb{C}\,;\, \mathrm{Im}(\xi) > 0\,\}$ representation; therefore, for the first part of the procedure, we shall work in this domain.
The Cayley transformation

$$\xi \;=\; -\,i\,\frac{z+1}{z-1} \qquad \longleftrightarrow \qquad z \;=\; \frac{\xi - i}{\xi + i} \tag{3.5}$$

maps the upper half plane $\mathbb{H}$ into the unit disk Δ and viceversa. Since its Schwarzian derivative vanishes, we have that

$$Q(z) \;=\; \big(\xi'(z)\big)^2\, \widetilde{Q}\big(\xi(z)\big) \;=\; -\,\frac{4}{(1-z)^4}\, \widetilde{Q}\big(\xi(z)\big)\;. \tag{3.6}$$

In the upper half plane representation, the condition of no energy momentum flow at infinity [14, 15] is that $\widetilde{T} \;=\; \overline{\widetilde{T}}$ on the real axis, which translates into

$$\widetilde{Q}(\xi) \;=\; \overline{\widetilde{Q}}(\xi) \qquad\quad \text{when} \qquad\quad \xi \in \mathbb{R} \tag{3.7}$$

and therefore, by analyticity, for all $\xi \in \mathbb{H}$. Instead, the condition of regularity of $\widetilde{Q}(\xi)$ at infinity is

$$\xi^4\, \widetilde{Q}(\xi) \;\sim\; O(1) \qquad\quad \text{when} \qquad\quad \xi \longrightarrow \infty\;. \tag{3.8}$$

Let us begin with the unperturbed case of a single source of finite charge η. Because of the $SL(2,\mathbb{R})$ invariance of the upper half plane, we can place this source in $\xi = i$.
The function $\widetilde{Q}_0(\xi)$ for the unperturbed case satisfying (3.7) can be written as

$$\widetilde{Q}_0(\xi) \;=\; \frac{1-\lambda_i^2}{4\,(\,\xi - i\,)^2} \;+\; \frac{1-\bar{\lambda}_i^2}{4\,(\,\xi + i\,)^2} \;+\; \frac{b_i}{2\,(\,\xi - i\,)} \;+\; \frac{\bar{b}_i}{2\,(\,\xi + i\,)}\;. \tag{3.9}$$

The complex numbers b_i and $\bar{b}_i = b_{-i}$ are the unperturbed accessory parameters related to the singularities in i and in its image $-i$, respectively. The parameter λ_i^2 is related to the charge η as follows

$$\eta(\eta-1) + \frac{1-\lambda_i^2}{4} \;=\; 0 \tag{3.10}$$

which tells us that $\lambda_i = \bar{\lambda}_i$, being $\eta \in \mathbb{R}$. Moreover, by imposing the regularity condition at infinity (3.8) for $\widetilde{Q}_0(\xi)$, we find

$$b_i \;=\; i\,2\eta(1-\eta) \tag{3.11}$$

and the expression of $\widetilde{Q}_0(\xi)$ becomes

$$\widetilde{Q}_0(\xi) \;=\; \frac{4\eta(\eta-1)}{(\xi^2+1)^2}\;. \tag{3.12}$$

In the Δ representation, it reads

$$Q_0(z) \;=\; \frac{\eta(1-\eta)}{z^2}\;. \tag{3.13}$$

Two independent solutions are $y_1(z) = z^\eta$ and $y_2(z) = z^{1-\eta}$ and their wronskian $\omega_{12} = y_1 y_2' - y_1' y_2 = 1-2\eta$ is constant. Except for a numerical factor, they correspond respectively to $\tilde{y}_1(\xi) = (1+i\xi)^\eta(1-i\xi)^{1-\eta}$ and $\tilde{y}_2(\xi) = (1-i\xi)^\eta(1+i\xi)^{1-\eta}$ in the $\mathbb{H}$ representation.

Now we perturb the previous geometry by introducing a new source at a generic point $\zeta \in \mathbb{H}$ with a small charge $\eta_2 = \varepsilon$.
We can write down the perturbed energy momentum tensor satisfying (3.7) as follows

$$\widetilde{Q}(\xi) \;=\; \widetilde{Q}_0(\xi) \,+\, \varepsilon\,\tilde{q}(\xi) \tag{3.14}$$

where $\widetilde{Q}_0(\xi)$ is the unperturbed energy momentum tensor (3.12) and the perturbation $\tilde{q}(\xi)$ is given by

$$\tilde{q}(\xi) \;=\; \frac{1}{(\,\xi-\zeta\,)^2} + \frac{1}{(\,\xi-\bar{\zeta}\,)^2} + \frac{\beta_i}{2\,(\,\xi-i\,)} + \frac{\bar{\beta}_i}{2\,(\,\xi+i\,)} + \frac{\beta_\zeta}{2\,(\,\xi-\zeta\,)} + \frac{\bar{\beta}_\zeta}{2\,(\,\xi-\bar{\zeta}\,)}\;. \tag{3.15}$$

Notice that now the accessory parameters are given by the sum of their unperturbed values, already determined, and a perturbation $O(\varepsilon)$, whose complex coefficients (i.e. β_i and β_ζ for the points i and ζ respectively) must satisfy the above mentioned conditions. The regularity condition for $\xi^3\tilde{q}(\xi) \to 0$ when $\xi \to \infty$ implies that

$$\begin{cases} \beta_i + \bar{\beta}_i \,+\, \left(\beta_\zeta + \bar{\beta}_\zeta\right) \;=\; 0 \\ 4 - i\left(\beta_\zeta + \bar{\beta}_\zeta\right) \,+ \zeta\,\beta_\zeta + \bar{\zeta}\,\bar{\beta}_\zeta \;=\; 0 \\ 4\left(\zeta+\bar{\zeta}\right) \,-\, \left(\beta_i + \bar{\beta}_i\right) + \zeta^2\,\beta_\zeta + \bar{\zeta}^2\,\bar{\beta}_\zeta \;=\; 0\;. \end{cases} \tag{3.16}$$

We can use a $SL(2,\mathbb{R})$ transformation which leaves i fixed to move the point ζ on the imaginary axis, $\zeta = i\tau$, with $\tau \in \mathbb{R}_0^+$ and $\tau \neq 1$. The system (3.16) simplifies to

$$\begin{cases} \mathrm{Re}(\beta_i) \;=\; \mathrm{Re}(\beta_{i\tau}) \;=\; 0 \\ \mathrm{Im}(\beta_i) \;=\; 2 \,-\, \tau\,\mathrm{Im}(\beta_{i\tau}) \;\equiv\; \beta \end{cases} \tag{3.17}$$

and we are left only with the parameter β to determine.
Through the transformation law (3.6), we can write the expression $q(z)$ of the perturbation in the Δ representation

$$q(z) = \frac{1}{(z-t)^2} + \frac{1}{(z-1/t)^2} - \frac{\beta}{z} + \left(\frac{2\,t+\beta}{1-t^2}\right)\frac{1}{z-t} - \left(t\,\frac{2+t\,\beta}{1-t^2}\right)\frac{1}{z-1/t} \tag{3.18}$$

where

$$t = \frac{\tau-1}{\tau+1} \in (-1,1)\setminus\{0\} \tag{3.19}$$

is the image in Δ of the point $i\tau \in \mathbb{H}$ through the Cayley transformation.

In the perturbed case, the conformal factor has the usual structure

$$\pi\mu b^2\, e^{\varphi_2(z)} = \frac{|\,\Omega_{12}|^2}{\left(Y_1(z)\,\overline{Y_1(z)} - Y_2(z)\,\overline{Y_2(z)}\,\right)^2} \tag{3.20}$$

where $\Omega_{12} = Y_1 Y_2' - Y_1' Y_2$.
The solutions $Y_j(z)$ of the perturbed problem can be written as a sum of the unperturbed solutions $y_j(z)$ and of a perturbation $O(\varepsilon)$ as follows

$$Y_i(z) = y_i(z) + \varepsilon\,\delta y_i(z) \qquad\qquad i = 1,\,2 \tag{3.21}$$

where $\delta y_j(z)$ satisfy the following inhomogeneous differential equation

$$\frac{d^2\delta y_i}{dz^2} + Q_0(z)\,\delta y_i = -\,q(z)\,y_i\;. \tag{3.22}$$

The solutions of this equation are given by the following integrals [5]

$$\begin{aligned} \delta y_i(z) &= -\frac{1}{\omega_{12}}\int_0^z dx\,\Big(y_1(x)\,y_2(z) - y_1(z)\,y_2(x)\Big)\,q(x)\,y_i(x) \\ &= -\frac{1}{\omega_{12}}\,I_{i1}(z)\,y_2(z) + \frac{1}{\omega_{12}}\,I_{i2}(z)\,y_1(z) \end{aligned} \tag{3.23}$$

where

$$I_{ij}(z) \equiv \int_0^z y_i(x)\,y_j(x)\,q(x)\,dx\;. \tag{3.24}$$

Notice that the integrals $I_{ij}(z)$ are invariant under the Cayley map

$$\tilde{I}_{ij}(\xi) = \int_i^\xi \tilde{y}_i(y)\,\tilde{y}_j(y)\,\tilde{q}(y)\,dy = \int_0^z y_i(x)\,y_j(x)\,q(x)\,dx = I_{ij}(z)\;. \tag{3.25}$$

Since we have chosen the position of the finite source as starting point, we have that $I_{ij}(0) = \tilde{I}_{ij}(i) = 0$.

More explicitly, the two independent solutions of the perturbed problem in terms of the integrals $I_{ij}(z)$ are

$$
\begin{aligned}
Y(z) \equiv \begin{pmatrix} Y_1(z) \\ Y_2(z) \end{pmatrix} &= \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} + \frac{\varepsilon}{\omega_{12}} \begin{pmatrix} I_{12}(z) & -I_{11}(z) \\ I_{22}(z) & -I_{12}(z) \end{pmatrix} \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} \\
&= \left(\mathbb{I} + \frac{\varepsilon}{\omega_{12}} M_t(z) \right) \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} .
\end{aligned} \tag{3.26}
$$

Notice that, since $\mathrm{tr} M_t(z) = 0$, then $\Omega_{12} = \omega_{12} + O(\varepsilon^2)$.
Moreover, if we consider a finite neighborhood of $z = 0$ not containing t and we let z to encircle once the origin, i.e. $z = \rho\, e^{i\varphi}$ with $0 < \rho < t$ with φ varying continuously from 0 to 2π, then the solutions $Y_1(z)$ and $Y_2(z)$ transform as follows

$$
Y_1(z) \longrightarrow e^{2\pi i\,\eta}\, Y_1(z) \qquad\qquad Y_2(z) \longrightarrow e^{2\pi i\,(1-\eta)}\, Y_2(z) . \tag{3.27}
$$

This ensures that the conformal factor is monodromic around the point $z = 0$.
Now the only freedom left for the vector $Y(z)$ is the multiplication by the matrix $K \in U(1,1)$

$$
Y(z) \longrightarrow K\, Y(z) \qquad\qquad K = \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix} \tag{3.28}
$$

where $k = 1 + \varepsilon\, h(t)$ with $h(t) \in \mathbb{C}$, i.e. the unperturbed value of k must be 1 in order to recover the classical solution (1.19), which describes correctly the geometry of the unperturbed case.
Now we have to impose the monodromy condition around the point $z = t$. To do this, we need to compute the change of $I_{ij}(z)$ when z is near the point t and turns once around it. From the expression of $q(z)$ given in (3.18), one easily sees that

$$
\begin{aligned}
\delta I_{ij}(t) &= \oint_t y_i(x)\, y_j(x)\, q(x)\, dx \\
&= 2\pi i \left(\frac{2\,t + \beta}{1 - t^2} \right) y_i(t)\, y_j(t) + 2\pi i\, \frac{d}{dz} \big(y_i(z)\, y_j(z) \big) \Big|_{z=t} .
\end{aligned} \tag{3.29}
$$

Thus, the transformation of the vector $Y(z)$ when one encircles $z = t$, including also the multiplication (3.28), is given by the following matrix

$$
\mathbb{I} + \frac{\varepsilon}{\omega_{12}} \begin{pmatrix} \delta I_{12}(t) & -\delta I_{11}(t)/k^2 \\ \delta I_{22}(t)/k^2 & -\delta I_{12}(t) \end{pmatrix} = \mathbb{I} + \frac{\varepsilon}{\omega_{12}} \begin{pmatrix} \delta I_{12}(t) & -\delta I_{11}(t) \\ \delta I_{22}(t) & -\delta I_{12}(t) \end{pmatrix} + O(\varepsilon^2) . \tag{3.30}
$$

The monodromy around t imposes the $U(1,1)$ nature of such a matrix; therefore, we must require that

$$
\overline{\delta I_{12}(t)} = -\,\delta I_{12}(t) \qquad\qquad \overline{\delta I_{22}(t)} = -\,\delta I_{11}(t) . \tag{3.31}
$$

From (3.29), we have that

$$\delta I_{12}(t) \;=\; 2\pi i \left(\frac{2\,t+\beta}{1-t^2}\; t + 1 \right) \tag{3.32}$$

$$\delta I_{11}(t) \;=\; 2\pi i\; t^{2\eta-1} \left(\frac{2\,t+\beta}{1-t^2}\; t + 2\,\eta \right) \tag{3.33}$$

$$\delta I_{22}(t) \;=\; 2\pi i\; t^{1-2\eta} \left(\frac{2\,t+\beta}{1-t^2}\; t + 2 - 2\,\eta \right) \,. \tag{3.34}$$

Since $\beta \in \mathbb{R}$, the first condition of (3.31) is already realized. Instead, the second condition provides the explicit expression of β

$$\beta \;=\; -\,2\; \frac{\eta + (1-\eta)\,t^2 - t^{2\,(1-2\eta)}\,(1-\eta+\eta\,t^2)}{t\,\big(1-t^{2\,(1-2\eta)}\big)} \,. \tag{3.35}$$

Thus, the reflection condition (3.7) together with the regularity requirement at infinity (3.8) and the monodromy of the perturbed solution (3.20) around 0 and t fix completely the perturbed accessory parameters.

Now, if we write $\varphi_2(z)$ in (3.20) as follows

$$\varphi_2(z) \;=\; \varphi_{cl}(z) + \epsilon\, \psi(z,t) + O(\epsilon^2) \tag{3.36}$$

where $\varphi_{cl}(z)$ is given by (1.19), by using the expression (3.26) for the perturbed solutions, we find that [5]

$$\begin{aligned} \psi(z,t\,) \;=\; -\,\frac{2}{w_{12}} \Bigg\{ & \frac{y_1\bar{y}_1 + y_2\bar{y}_2}{y_1\bar{y}_1 - y_2\bar{y}_2} \Big(I_{12} + \bar{I}_{12} + 2\,\mathrm{Re}\, h(t) \Big) \\ & - \frac{\bar{y}_1 y_2\, I_{11} + y_1\bar{y}_2\, I_{22}}{y_1\bar{y}_1 - y_2\bar{y}_2} - \frac{y_1\bar{y}_2\, \bar{I}_{11} + \bar{y}_1 y_2\, \bar{I}_{22}}{y_1\bar{y}_1 - y_2\bar{y}_2} \Bigg\} \,. \end{aligned} \tag{3.37}$$

The parameter $\mathrm{Re}\, h(t)$ appearing in this expression cannot be determined through monodromy arguments because the term it multiplies

$$f(z) \;\equiv\; \frac{y_1\bar{y}_1 + y_2\bar{y}_2}{y_1\bar{y}_1 - y_2\bar{y}_2} \tag{3.38}$$

is a monodromic solution of the homogeneous differential equation

$$-\,\partial_z\partial_{\bar{z}}\, f(z) + 2\pi\,\mu b^2\, e^{\varphi_{cl}} f(z) \;=\; 0 \,. \tag{3.39}$$

Instead, $\mathrm{Re}\, h(t)$ is fixed through the analysis of the behavior of φ_2 when $|z| \to 1$. Indeed, $f(z)$ violates the asymptotic (1.12) because it diverges as $O\big(1/(1-z\bar{z})\big)$ when $|z| \to 1$;

therefore $\mathrm{Re}\, h(t)$ is uniquely determined by imposing the boundary conditions (1.12) for the classical field φ_2. Since the leading logarithmic divergence in (1.12) is already recovered by φ_{cl}, then $\psi(z,t)$ must not diverge when $|z| \to 1$.
Before computing $\psi(z,t)$ explicitly and examining its boundary behavior, we notice that $\psi(z,t)$ provides also the Green function on the classical background with one finite source, i.e. φ_{cl}. Indeed, since φ_2 describes the classical background of the pseudosphere with one finite source of charge $\eta_1 = \eta$ in $z_1 = 0$ and another source of infinitesimal charge $\eta_2 = \epsilon$ placed in $z_2 = t$, it satisfies the following Liouville equation

$$-\,\partial_z \partial_{\bar z} \varphi_2 + 2\pi\mu b^2\, e^{\varphi_2} \;=\; 2\pi\, \eta\, \delta^2(z) + 2\pi\, \epsilon\, \delta^2(z-t\,)\;. \tag{3.40}$$

Taking the derivative of this equation w.r.t. ϵ and setting $\epsilon = 0$, we find that $\psi(z,t)$ solves the following equation

$$-\,\partial_z \partial_{\bar z}\, \psi + 2\pi\mu b^2\, e^{\varphi_{cl}} \psi \;=\; 2\pi\, \delta^2(z-t\,) \tag{3.41}$$

and therefore the Green function $g(z,t)$ arising from the quadratic part of the quantum action (1.18) is given by

$$g(z,t) \;=\; \langle\, \chi(z)\chi(t)\,\rangle \;=\; \frac{1}{4}\, \psi(z,t)\;. \tag{3.42}$$

To understand better the final expression for $g(z,t)$ it more useful to write it in the form given below

$$\begin{aligned} g(z,t) \;=\; -\,\frac{1}{2w_{12}} \Bigg\{ &\frac{y_1\bar y_1 + y_2 \bar y_2}{y_1 \bar y_1 - y_2 \bar y_2} \left(I_{12} + \bar I_{12} + 2\,\mathrm{Re}\, h(t) \right) \\ &- \frac{y_1 \bar y_1}{y_1\bar y_1 - y_2 \bar y_2} \left(\frac{y_2}{y_1}\, I_{11} + \frac{\bar y_2}{\bar y_1}\, \bar I_{11} \right) - \frac{y_2 \bar y_2}{y_1\bar y_1 - y_2 \bar y_2} \left(\frac{y_1}{y_2}\, I_{22} + \frac{\bar y_1}{\bar y_2}\, \bar I_{22} \right) \Bigg\}\;. \end{aligned} \tag{3.43}$$

As noticed before and computed in appendix B, $\mathrm{Re}\, h(t)$ is fixed by the asymptotic behavior of $g(z,t)$ when $|z| \longrightarrow 1$ and the result is

$$\mathrm{Re}\, h(t) \;=\; \frac{1}{2} \left(\frac{1+t^{2\,(1-2\eta)}}{1-t^{2\,(1-2\eta)}}\, \log t^{2\,(1-2\eta)} + 2 \right)\;. \tag{3.44}$$

Moreover, by exploiting the invariance under rotation, one can easily generalize all these expressions to a complex $t \in \Delta$.

The Green function in the explicit symmetric form is given by

$$g(z,t) \;=\; -\frac{1}{2}\,\frac{1+(z\bar z)^{1-2\eta}}{1-(z\bar z)^{1-2\eta}}\,\frac{1+(t\bar t)^{1-2\eta}}{1-(t\bar t)^{1-2\eta}}\,\log\omega(z,t) \;-\; \frac{1}{1-2\eta} \tag{3.45}$$

$$-\frac{1}{1-(z\bar z)^{1-2\eta}}\,\frac{1}{1-(t\bar t)^{1-2\eta}}\,\Big\{(z\bar t)^{1-2\eta}\Big(B_{z/t}(2\eta,\,0)-B_{z\bar t}(2\eta,\,0)\Big)$$

$$+\,(\bar z t)^{1-2\eta}\Big(B_{t/z}(2\eta,\,0)-B_{1/(z\bar t)}(2\eta,\,0)\Big)+\text{c.c.}\Big\}$$

where $\omega(z,t)$ is the $SU(1,1)$ invariant

$$\omega(z,t) \;=\; \left|\,\frac{z-t}{1-z\,\bar t}\,\right|^2 \tag{3.46}$$

which is related to the geodesic distance on the pseudosphere without sources.
Notice that only the special case $B_x(a,0)$ of the incomplete beta function $B_x(a,b)$ occurs; it is related to the hypergeometric function $F(a,1;a+1;\,x)$ as follows

$$B_x(a,0) \;=\; \frac{x^a}{a}\,F(a,1;a+1;\,x) \;=\; \int_0^x \frac{y^{a-1}}{1-y}\,dy \;=\; \sum_{n\geqslant 0}\frac{x^{a+n}}{a+n}\;. \tag{3.47}$$

In appendix B it is shown that $g(z,t)$ is regular at the origin and, through a partial wave expansion, it is also shown that

$$g(z,t) \;=\; O\big((1-z\bar z)^2\big) \qquad \text{when} \qquad |z|\to 1\;. \tag{3.48}$$

Because of this asymptotic at infinity, the quantum action (1.18) becomes

$$S_q[\,\chi\,] \;=\; \int_\Delta\Big(\frac{1}{\pi}\,\partial_z\chi\,\partial_{\bar z}\chi+2\mu b^2\,e^{\varphi_{cl}}\,\chi^2\Big)\,d^2z \;+\; \sum_{k\geqslant 3}\frac{(2b)^k}{k!}\int_\Delta \mu\,e^{\varphi_{cl}}\,\chi^k\,d^2z \tag{3.49}$$

where φ_{cl} is given by (1.19).
A related function that will play a crucial role in what follows is the Green function (3.45) at coincident points regularized according to the ZZ procedure, i.e. (2.4).
By computing (2.4) or using the series representation (B.10) of the hypergeometric function $F(a,1;1+a;\,x)$, we find that $g(z,z)$ is given by

$$g(z,z) \;=\; \left(\frac{1+(z\bar z)^{1-2\eta}}{1-(z\bar z)^{1-2\eta}}\right)^2\log(1-z\bar z) \;-\; \frac{1}{1-2\eta}\,\frac{1+(z\bar z)^{1-2\eta}}{1-(z\bar z)^{1-2\eta}} \tag{3.50}$$

$$+\,\frac{2\,(z\bar z)^{1-2\eta}}{\big(1-(z\bar z)^{1-2\eta}\big)^2}\,\Big(B_{z\bar z}(2\eta\,,0)+B_{z\bar z}(2-2\eta\,,0)$$

$$+\;2\gamma_{\scriptscriptstyle E}+\psi(2\eta)+\psi(2-2\eta)-\log z\bar z\Big)$$

where γ_{E} is the Euler constant and $\psi(x) = \Gamma'(x)/\Gamma(x)$.
The asymptotic behavior at infinity (i.e. when $|z| \to 1$) of $g(z,z)$ is the following

$$g(z,z) \;=\; \log(1-z\bar{z}) - 1 - \frac{\eta(1-\eta)}{6}\,(1-z\bar{z})^2 + O\big((1-z\bar{z})^3\big)\;. \tag{3.51}$$

Notice that the dependence on the charge η occurs only at $O\big((1-z\bar{z})^2\big)$.

4 The one point function: the quantum determinant

In this section we compute the quantum determinant for $N = 1$ explicitly and we compare this result with the corresponding order in the expansion of the one point function obtained in the bootstrap approach.
To compute the quantum determinant for the one point function, we apply the formula (2.2) with $\varphi_{B} = \varphi_{cl}$ and with $g(z,z)$ given by (3.50), i.e. for a single source $\eta_1 = \eta$ at $z_1 = 0$.
To perform this integral, it is more convenient to adopt the variable $u \equiv (z\bar{z})^{1-2\eta}$ in the radial integration. Then, after an integration by parts, we obtain

$$\frac{\partial}{\partial\eta}\,\log\big(\,\mathrm{Det}\,D(\eta,0)\,\big)^{-1/2} \;=\; 2\,\gamma_{E} \;+\; 2\,\psi(1-2\eta) \;+\; \frac{3}{1-2\eta}\;. \tag{4.1}$$

Integrating back in η with the initial condition

$$\log\big(\,\mathrm{Det}\,D(\eta,0)\,\big)^{-1/2}\Big|_{\eta=0} \;=\; 0 \tag{4.2}$$

we find the explicit expression of the logarithm of the quantum determinant

$$\log\big(\,\mathrm{Det}\,D(\eta,0)\,\big)^{-1/2} \;=\; 2\eta\,\gamma_{E} - \log\Gamma(1-2\eta) - \frac{3}{2}\,\log(1-2\eta)\;. \tag{4.3}$$

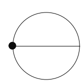
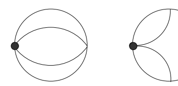
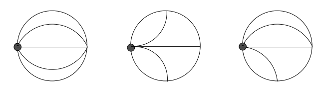
Putting this result together with the classical contribution (1.21), we have the first two terms of the perturbative expansion in the coupling constant b of the one point function

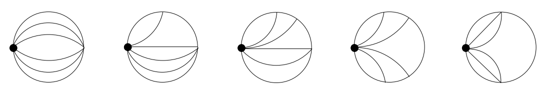
$$\begin{aligned}\big\langle\, V_{\eta/b}(0)\,\big\rangle \;&=\; \big\langle\, e^{2(\eta/b)\phi(0)}\,\big\rangle \;=\; \exp\left\{-\frac{1}{b^2}\,\Big[\,\eta\,\log(\pi\mu b^2) + 2\eta + (1-2\eta)\,\log(1-2\eta)\,\Big]\right\}\\ &\times\,\frac{e^{2\eta\gamma_E}}{\Gamma(1-2\eta)\,(1-2\eta)^{3/2}}\,\big(\,1+O(b^2)\,\big)\;.\end{aligned} \tag{4.4}$$

The expansion (4.4) agrees with the expansion in the coupling constant of the logarithm of the formula $U_{1,1}(\eta/b)$ given in (1.23), found by ZZ [1] through the bootstrap method.

We remark that the result (4.4) corresponds to the summation of two infinite classes of perturbative graphs computed on the regular background, i.e. with the classical field given by $\varphi_{cl}|_{\eta=0}$ and the propagator $\hat{g}(z,z')$ written in (B.22). These graphs can be recovered by expanding the logarithm of (4.4) in $\eta = \alpha\, b$.
The classical part gives rise to the following series of graphs

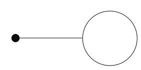
$$-\frac{1}{b^2}\left[\eta\,\log(\pi\mu b^2) + 2\eta + (1-2\eta)\,\log(1-2\eta)\right] \;=\; \frac{\eta}{b^2}\,\varphi_{cl}(0)|_{\eta=0} - 2\,\frac{\eta^2}{b^2} \tag{4.5}$$

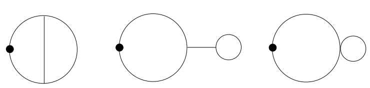
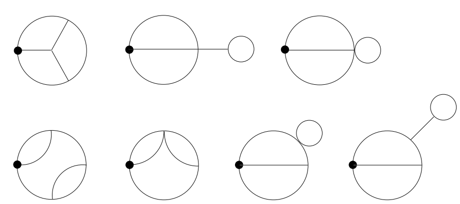
$$+\frac{\eta^3}{b^2}\Big\{$$  $$\Big\} + \frac{\eta^4}{b^2}\Big\{$$  $$\Big\} + \frac{\eta^5}{b^2}\Big\{$$  $$\Big\}$$

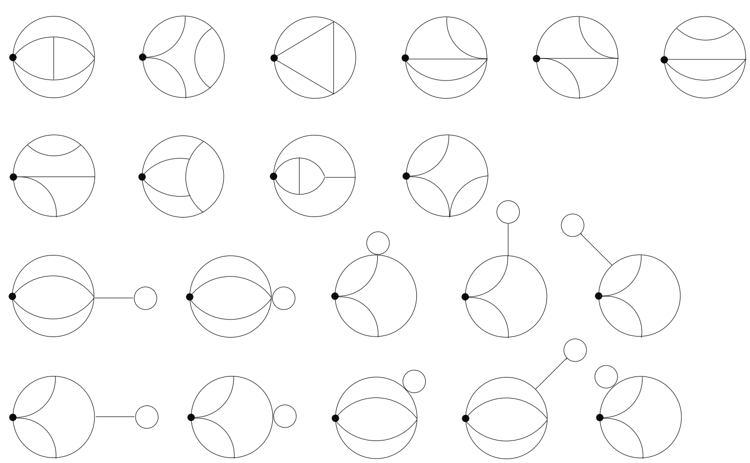
$$+\frac{\eta^6}{b^2}\Big\{$$  $$\Big\} + \ldots$$

$$= \; -\frac{\eta}{b^2}\,\log(\pi\mu b^2) \;-\; 2\,\frac{\eta^2}{b^2} \;-\; \frac{4}{3}\,\frac{\eta^3}{b^2} \;-\; \frac{4}{3}\,\frac{\eta^4}{b^2} \;-\; \frac{8}{5}\,\frac{\eta^5}{b^2} \;-\; \frac{32}{15}\,\frac{\eta^6}{b^2} \;+\;\ldots$$

while the quantum determinant contribution contains the following perturbative orders

$$2\,\gamma_E\,\eta - \log\Gamma(1-2\eta) - \frac{3}{2}\,\log(1-2\eta) \;=\; \eta\,\Big\{$$  $$\Big\}$$

$$+\,\eta^2\,\Big\{$$  $$\Big\} + \eta^3\,\Big\{$$  $$\Big\}$$

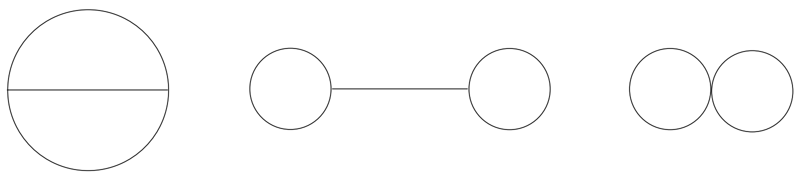
$$+\,\eta^4\,\Big\{$$  $$\Big\} + \ldots \tag{4.6}$$

$$= \; 3\,\eta + \left(3 - \frac{\pi^2}{3}\right)\eta^2 + \frac{4}{3}\big(3 - 2\,\zeta(3)\big)\,\eta^3 + 2\left(3 - \frac{\pi^4}{45}\right)\eta^4 + \ldots$$

All the propagators in the figures are given by $\hat{g}(z,z')$. Those starting from the source

were denoted in [4] by dotted lines because they still represent a classical field even though computationally they are given by the same expression.
The first orders of the classical part have been determined in [1] while the orders $O(\eta^4/b^2)$ and $O(\eta^5/b^2)$ have been computed in [4]. As for the quantum determinant, the $O(\eta^2)$ contribution agrees with the result obtained by ZZ [1], while the $O(\eta^3)$ term agrees with the explicit, but far more difficult computation performed in [8]. Instead, the $O(\eta^4)$ term and the further orders in the quantum determinant are new results, obtained as byproducts of the knowledge of the quantum determinant for every value of $\eta < 1/2$.

With some effort, one could compute the $O(b^2)$ contribution in (4.4) within our framework. It is given by the following three graphs



(A) (B) (C)

where the propagator is given by (3.45).

5 The two point function

In this section we apply the technique developed in the previous sections to compute the following two point function on the pseudosphere

$$\left\langle V_{\eta/b}(z_1)\, V_{\varepsilon/b}(z_2) \right\rangle \tag{5.1}$$

up to $O(\varepsilon)$ and $O(b^0)$ included, but to all orders in η and in the invariant distance.
According to [1], this result is related to the conformal block with null intermediate dimension through to the "boundary" representation of the "normalized" two point function

$$g_{\eta/b,\,\varepsilon/b}(\omega) \equiv \frac{\left\langle V_{\eta/b}(z_1)\, V_{\varepsilon/b}(z_2) \right\rangle}{\left\langle V_{\eta/b}(z_1) \right\rangle \left\langle V_{\varepsilon/b}(z_2) \right\rangle} = (1-\omega)^{2\Delta_{\eta/b}}\, \mathcal{F}\left(\begin{matrix} \eta/b & \varepsilon/b \\ \eta/b & \varepsilon/b \end{matrix} \,;\, iQ/2,\, 1-\omega \right) \tag{5.2}$$

where $\omega(z_1, z_2)$ is the $SU(1,1)$ invariant given in (3.46).
The procedure will be to compute the classical action and the quantum determinant on the background (3.36) describing the pseudosphere with two curvature singularities: a finite one $\eta_1 = \eta$ in $z_1 = 0$ and an infinitesimal one $\eta_2 = \varepsilon$ in $z_2 = t$. Since this classical background is known up to $O(\varepsilon)$, our results will be exact in η and perturbative in ε up

to $O(\varepsilon)$ included. This perturbative background has been already computed in section 3 and it is given by

$$\varphi_2(z) \;=\; \varphi_{cl}(z) + 4\,\epsilon\, g(z,t) + O(\epsilon^2) \tag{5.3}$$

where $\varphi_{cl}(z)$ is the background field (1.19) describing the pseudosphere with a single finite source $\eta_1 = \eta$ placed in $z_1 = 0$ and $g(z,t)$ is the propagator (3.45).
The two point function (5.1) up to $O(b^2)$ is

$$\begin{aligned}
\left\langle\, V_{\eta/b}(0)\, V_{\varepsilon/b}(t)\,\right\rangle \;&=\; e^{-S_{cl}(\eta,\,0;\,\varepsilon,\,t)+S_{cl}(0)} \;\times \qquad (5.4)\\
&\times\, \frac{1}{Z_0}\int \mathcal{D}\,[\,\chi\,]\;\exp\left\{-\,\frac{1}{2}\int_\Delta \chi\left(-\,\frac{2}{\pi}\,\partial_z\partial_{\bar z} \,+\, 4\mu b^2\, e^{\varphi_2}\right)\chi\, d^2 z\right\}\big(\,1+O(b^2)\,\big)\\
&=\; e^{-S_{cl}(\eta,\,0;\,\varepsilon,\,t)+S_{cl}(0)} \;\times\\
&\times\;\big(\,\mathrm{Det}\, D(\eta,\,0)\,\big)^{-1/2}\left(1 \,-\, 8\,\mu b^2 \varepsilon \int_\Delta g(z,t)\, e^{\varphi_{cl}(z)} g(z,z)\, d^2 z \,+ O(\varepsilon^2)\right)\big(\,1+O(b^2)\,\big)
\end{aligned}$$

where $S_{cl}(\eta,0;\varepsilon,t)$ is the classical action (1.9) evaluated on the field $\varphi_2(z)$, while $S_{cl}(0)$ and Z_0 are respectively the classical contribution and the quadratic part of the partition function Z occurring in (1.1) and (1.2).
The denominator occurring in (5.2) with $z_1 = 0$ and $z_2 = t$ up to $O(b^2)$ reads

$$e^{-S_{cl}(\eta,\,0)+S_{cl}(0)}\big(\mathrm{Det}\, D(\eta,\,0)\big)^{-1/2}\; e^{-S_{cl}(\varepsilon,\,t)+S_{cl}(0)}\big(\mathrm{Det}\, D(\varepsilon,\,t)\big)^{-1/2}\,\big(\,1+O(b^2)\,\big) \tag{5.5}$$

where $\big(\mathrm{Det}\, D(\eta,\,0)\big)^{-1/2}$ has been already computed and it is given by (4.3).
Evaluating the classical action (1.9) on the field (5.3), we get the following perturbative expression in ε

$$S_{cl}(\eta,0;\,\varepsilon,t) \;=\; S_{cl}(\eta,0) - \frac{\varepsilon}{b^2}\,\varphi_{cl}(t) + O(\varepsilon^2)\;. \tag{5.6}$$

This formula for $\eta = 0$ provides

$$S_{cl}(\varepsilon,t) \;=\; S_{cl}(0) - \frac{\varepsilon}{b^2}\,\varphi_{cl}(t)|_{\,\eta\,=\,0} + O(\varepsilon^2) \tag{5.7}$$

while

$$\big(\mathrm{Det}\, D(\varepsilon,t)\big)^{-1/2} \;=\; 1 -\, 8\,\mu b^2\varepsilon \int_\Delta \hat g(z,t)\, e^{\,\varphi_{cl}(t)|_{\,\eta\,=\,0}}\, \hat g(z,z)\, d^2 z \,+ O(\varepsilon^2) \tag{5.8}$$

which is the quantum determinant contribution occurring in (5.4) evaluated on $\eta = 0$.
Thus, to orders $O(\varepsilon)$ and $O(b^0)$ included, the logarithm of the “normalized” two point function (5.2) with $z_1 = 0$ and $z_2 = t$ becomes

$$\begin{aligned}
\log\, \frac{\left\langle\, V_{\eta/b}(0)\, V_{\varepsilon/b}(t)\,\right\rangle}{\left\langle\, V_{\eta/b}(0)\,\right\rangle\left\langle\, V_{\varepsilon/b}(t)\,\right\rangle} \;&=\; \frac{\varepsilon}{b^2}\left\{\varphi_{cl}(t) - \varphi_{cl}(t)|_{\,\eta\,=\,0}\right\} \qquad (5.9)\\
&-\,8\,\mu b^2 \varepsilon\left\{\int_\Delta g(z,t)\, e^{\varphi_{cl}(z)} g(z,z)\, d^2 z \,-\, \int_\Delta \hat g(z,t)\, e^{\,\varphi_{cl}(t)|_{\,\eta\,=\,0}}\, \hat g(z,z)\, d^2 z\right\}.
\end{aligned}$$

The first integral occurring in this expression can be computed by exploiting the partial wave representation given in appendix B (see (B.26) and (B.27)). Because of invariance under rotations, only the wave $m = 0$ contributes and we have

$$- 8\,\mu b^2 \varepsilon \int_\Delta g(z,t)\, e^{\varphi_{cl}(z)} g(z,z)\, d^2 z \;\; = \tag{5.10}$$

$$- 8\,\mu b^2 \pi\, \varepsilon \left\{ b_0(|t|^2) \int_0^{|t|^2} a_0(|z|^2)\, e^{\varphi_{cl}(z)} g(z,z)\, d|z|^2 + a_0(|t|^2) \int_{|t|^2}^1 b_0(|z|^2)\, e^{\varphi_{cl}(z)} g(z,z)\, d|z|^2 \right\}$$

where $g(z,z)$ is given in (3.50). These integrals can be computed explicitly through an integration by parts, while the other integral occurring in (5.9) is the tadpole contribution in the geometry of the pseudosphere without singularities [1].
We remark that (5.9) is invariant under $SU(1,1)$ transformations, as expected. Indeed, since both classical fields $\varphi_{cl}(t)$ and $\varphi_{cl}(t)|_{\eta=0}$ transform as in (1.14) under $SU(1,1)$ and dw/dz is independent of η, the classical term is invariant. As for the quantum determinant contribution, by using the transformation law (2.8) for $g(z,z)$ and $\hat{g}(z,z)$, together with the equations for the Green functions $g(z,t)$ and $\hat{g}(z,t)$, we find that the variation under $SU(1,1)$ of the quantum determinant contribution is given by

$$\lim_{r\to 1} \frac{2}{2\pi i} \left[\oint_{\partial\Delta_r} \partial_z \big(g(z,t) - \hat{g}(z,t) \big) \log \left| \frac{dw}{dz} \right|^2 dz + \oint_{\partial\Delta_r} \big(g(z,t) - \hat{g}(z,t) \big) \log \frac{d\bar{w}}{d\bar{z}}\, d\bar{z} \right] . \tag{5.11}$$

Since $g(z,t) - \hat{g}(z,t) \;=\; O\big((1 - z\bar{z})^2\big)$ when $|z| \to 1$, these integrals vanish in the limit $r \to 1$; hence the expression (5.9) is invariant under $SU(1,1)$ transformations and we can substitute $t\bar{t}$ with the $SU(1,1)$ invariant ratio ω in the explicit expression for (5.9).
Thus, including also the classical terms, (5.9) becomes

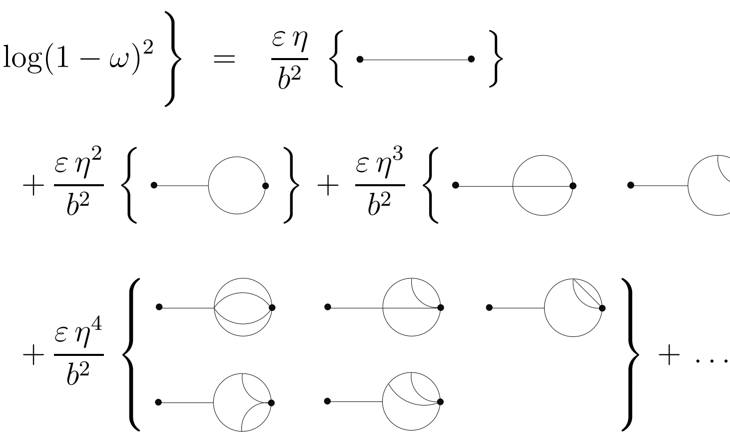
$$\begin{aligned} \log \frac{\langle V_{\eta/b}(z_1)\, V_{\varepsilon/b}(z_2) \rangle}{\langle V_{\eta/b}(z_1) \rangle \langle V_{\varepsilon/b}(z_2) \rangle} \;\; &= \;\; \frac{\varepsilon}{b^2} \left\{ - \log \frac{\big(\omega^\eta - \omega^{1-\eta}\big)^2}{(1-2\eta)^2} \; + \; \log(1-\omega)^2 \right\} \\ &+ \varepsilon \left\{ \frac{2}{\big(1 - \omega^{1-2\eta}\big)^2} \Bigg(B_\omega(2-2\eta,0) + \psi(2-2\eta) + \gamma_{\scriptscriptstyle E} + \frac{1}{2(1-2\eta)} \right. \\ &\quad + \omega^{2(1-2\eta)} \left(B_\omega(2\eta,0) + \psi(2\eta) + \gamma_{\scriptscriptstyle E} + \frac{3}{2(1-2\eta)} \; - \log\omega \right) \\ &\quad \left. + 2\,\omega^{1-2\eta} \left(\log(1-\omega) - \frac{1}{1-2\eta} \right) \Bigg) + \; 2\, \log(1-\omega) \; - \; 3 \right\} . \end{aligned} \tag{5.12}$$

with $\omega = \omega(z_1, z_2)$ given by (3.46). According to the analysis performed in [1], up to the orders $O(b^0)$ and $O(\varepsilon)$ included, but exact in η and $\omega(z_1, z_2)$, the formula (5.12) gives the

expansion of the logarithm of the conformal block occurring in (5.2).

The expression (5.12), which is exact in η and ω up to orders $O(b^0)$ and $O(\varepsilon)$ included, provides the summation of two infinite classes of graphs, ordered according to a power expansion in η.
In [4] the classical part and the quantum determinant were computed respectively up to $O(\varepsilon\eta^3/b^2)$ and $O(\varepsilon\eta)$ included, by the explicit computation of every single graph. The procedure presented here extends largely the results obtained in [4] because it allows to get directly the sum of infinite classes of graphs, from which one can find the contribution of all the graphs occurring at a given perturbative order without computing them separately.
The classical part at $O(\varepsilon)$ gives

$$\frac{\varepsilon}{b^2}\left\{-\log\frac{\left(\omega^\eta-\omega^{1-\eta}\right)^2}{(1-2\eta)^2}+\log(1-\omega)^2\right\} \quad = \tag{5.13}$$



$$= 4\,\frac{\varepsilon\,\eta}{b^2}\,\hat{g}(z_1,z_2)+4\,\frac{\varepsilon\,\eta^2}{b^2}\left(\frac{2\,\omega\,\log\omega}{(1-\omega)^2}-1\right)-\frac{8}{3}\,\frac{\varepsilon\,\eta^3}{b^2}\left(\frac{3\,\omega\,(1+\omega)\,\log\omega}{(1-\omega)^3}+2\right)$$
$$+\frac{4}{3}\,\frac{\varepsilon\,\eta^4}{b^2}\left(\frac{4\,\omega\,(1+4\,\omega+\omega^2)\,\log\omega}{(1-\omega)^4}-6\right)+\ldots \tag{5.14}$$

while the quantum determinant contribution at $O(\varepsilon)$ provides the following infinite class

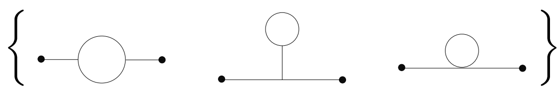
of perturbative graphs

$$
\varepsilon \left\{ \frac{2}{\left(1-\omega^{1-2\eta}\right)^2} \left(B_\omega(2-2\eta,0) + \psi(2-2\eta) + \gamma_{\scriptscriptstyle E} + \frac{1}{2(1-2\eta)} \right.\right. \tag{5.15}
$$

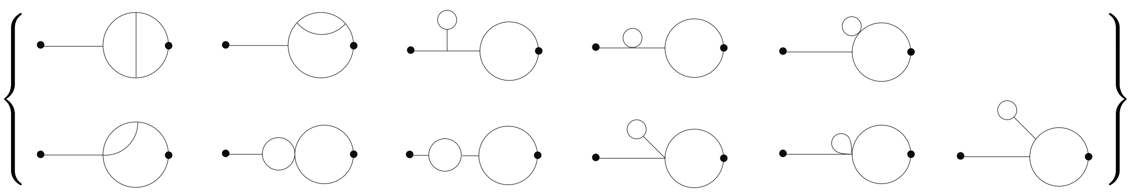
$$
+\,\omega^{2(1-2\eta)} \left(B_\omega(2\eta,0) + \psi(2\eta) + \gamma_{\scriptscriptstyle E} + \frac{3}{2(1-2\eta)} - \log\omega \right)
$$

$$
\left.\left. +\,2\,\omega^{1-2\eta} \left(\log(1-\omega) - \frac{1}{1-2\eta} \right) \right) + 2\,\log(1-\omega)\,-\,3 \right\} \; =
$$

$= \;\varepsilon\,\eta$



$+\,\varepsilon\,\eta^2$



$+ \ldots$

$$
= \;\; 2\,\varepsilon\,\eta \left(3 \,+\, \frac{2\,\omega^2 \log\omega}{(1-\omega)^2} \,-\, 2\,\frac{1+\omega}{1-\omega}\,\mathrm{Li}_2(1-\omega) \right)
$$

$$
+\,\frac{4\,\varepsilon\,\eta^2}{(1-\omega)^2} \left(3\,-2\left(1+\omega^2\right)\zeta(3)\,+\,\frac{2}{3}\,\pi^2\,\omega\,\log\omega\,-\,\frac{\omega^2\,(5+\omega)\,\log\omega}{(1-\omega)} \right.
$$

$$
\left. -\;2\left(1+4\,\omega+\omega^2\right)\,\log(1-\omega)\,\log\omega\,-\,2\,(1+\omega)^2\log\omega\,\mathrm{Li}_2(\omega)\,+\,2\left(1+\omega^2\right)\mathrm{Li}_3(\omega) \right)
$$

$+ \ldots$

where $\mathrm{Li}_\nu(x)$ is the polylogarithm function.

Notice that, by using (B.10), the behavior of (5.12) when $\omega(z_1,z_2)\to 1$ is

$$
\log\,\frac{\left\langle\, V_{\eta/b}(z_1)\,V_{\varepsilon/b}(z_2)\,\right\rangle}{\left\langle\, V_{\eta/b}(z_1)\,\right\rangle\left\langle\, V_{\varepsilon/b}(z_2)\,\right\rangle} \;=\; \left(\frac{\eta\,(1-\eta)}{3}\,\frac{\varepsilon}{b^2} \,-\, \frac{\eta\,(1-7\eta)}{18}\,\varepsilon \right)(1-\omega)^2 + O\big((1-\omega)^3\big) \tag{5.16}
$$

i.e. $g_{\,\eta/b,\,\varepsilon/b}(\omega)\to 1$ for the normalized two point function (5.2).
The fact that $\big\langle V_{\eta/b}(z_1)\,V_{\varepsilon/b}(z_2)\big\rangle \;\to\; \big\langle V_{\eta/b}(z_1)\big\rangle\,\big\langle V_{\varepsilon/b}(z_2)\big\rangle$ when $\omega(z_1,z_2)\;\to\;1$, i.e. when the geodesic distance diverges, is the cluster property and it is the boundary condition used in the bootstrap approach [1] to characterize the pseudosphere.

As a further check of our result (5.12), we consider the auxiliary bulk two point function $\langle V_{-1/(2b)}(z_1)\, V_{\varepsilon/b}(z_2)\rangle$ containing the primary field $V_{-1/(2b)}(z_1)$, which is degenerate at level 2. When the two point function contains a degenerate primary field at level 2, it satisfies a second order linear differential equation and it can be determined explicitly [15]. In our case, we have that [2]

$$g_{-1/(2b),\,\varepsilon/b}(\omega) \;=\; \omega^{\varepsilon/b^2}\;{}_2F_1\big(\,1+1/b^2,\, 2\varepsilon/b^2;\, 2+2/b^2;\, 1-\omega\,\big) \tag{5.17}$$

Expanding the logarithm of this expression, we find

$$\log\big[\, g_{-1/(2b),\,\varepsilon/b}(\omega)\,\big] \;=\; \frac{\varepsilon}{b^2}\,\big(\, \log\omega \;+\; 2\,\log 2 \;-\; 2\,\log(1+\omega)\,\big) + \frac{\varepsilon}{2}\,\left(\frac{1-\omega}{1+\omega}\right)^2 + O(\varepsilon b^2) \tag{5.18}$$

up to $O(\varepsilon)$ and $O(b^0)$ included. The expansion (5.18) agrees with our expansion (5.12) evaluated for $\eta = -1/2$.

Conclusions

We have obtained the one and two point functions on the pseudosphere with heavy charges to one loop. For the one point function agreement is found with the bootstrap formula given by ZZ [1] while the two point function provides a new expression for the case of one finite charge and an infinitesimal one.
Furthermore we have proved that the correct quantum dimensions recovered to one loop are left unchanged to all orders perturbation theory.
In the next publication [18] we extend the present approach to the conformal boundary case.

Appendices

A The background field at infinity

In this appendix we examine the behavior of the classical background on the pseudosphere at infinity in presence of N sources. We already saw in section 1 that the one source solution (1.19) behaves at infinity like

$$\varphi_{cl}(z) \;=\; -\log(1-z\bar z)^2 + \text{const} + O\big((1-z\bar z)^2\big) \tag{A.1}$$

i.e. no $O(1-z\bar{z})$ term occurs. This is relevant to have the second line of (1.10) vanishing for a quantum field χ which behaves like $\log(1-z\bar{z})$ when $|z|\to 1$. Here we prove that the term $O(1-z\bar{z})$ is absent also in the background φ_B generated by N sources. Since it is simpler to work in the $\mathbb{H}$ representation, we begin by using it.
Being $\widetilde{Q}(\xi)$ a real function, we can choose two real independent solutions $\tilde{y}_j(\xi)$ of

$$\tilde{y}_j''(\xi)+\widetilde{Q}(\xi)\tilde{y}_j(\xi)=0 \qquad\qquad j\,=\,1,2\,. \tag{A.2}$$

Using the fact the the wronskian is different from zero, it is simple to prove the following lemma: the identical vanishing of $a\tilde{y}_1^2+b\tilde{y}_2^2+c\tilde{y}_1\tilde{y}_2$ implies $a=b=c=0$. As explained in section 1, the solution of the Liouville equation is given by

$$\pi\mu b^2 e^{\tilde{\varphi}_B(\xi)}=\frac{|w_{12}|^2}{F^2} \tag{A.3}$$

where the most general form for F is

$$F\;=\;\big(\alpha\,\tilde{y}_1(\xi)+\beta\,\tilde{y}_2(\xi)\big)\big(\bar{\alpha}\,\tilde{y}_1(\bar{\xi})+\bar{\beta}\,\tilde{y}_2(\bar{\xi})\big)\;-\;\big(\gamma\,\tilde{y}_1(\xi)+\delta\,\tilde{y}_2(\xi)\big)\big(\bar{\gamma}\,\tilde{y}_1(\bar{\xi})+\bar{\delta}\,\tilde{y}_2(\bar{\xi})\big) \tag{A.4}$$

The identical vanishing of F for real ξ implies, as a consequence of the previous lemma, $\gamma=\bar{\alpha}$ and $\delta=\bar{\beta}$, hence we can write

$$F=Y(\xi)\overline{Y}(\bar{\xi})-\overline{Y}(\xi)Y(\bar{\xi})\,. \tag{A.5}$$

We notice that F is odd in $\mathrm{Im}\xi$, so that for small $\mathrm{Im}\xi$ we have

$$\frac{1}{F^2}\;=\;\frac{1}{4\big(\,\mathrm{Im}\xi+O\big((\mathrm{Im}\xi)^3\big)\big)^2}\;=\;\frac{1}{4(\mathrm{Im}\xi)^2}\;+\;O(1)\,. \tag{A.6}$$

On the other hand, if we start from the Δ representation, being w_{12} invariant, for $z\bar{z}\to 1$ we have

$$\begin{aligned}\pi\mu b^2 e^{\varphi_B(z)} \;&=\; \frac{|w_{12}|^2}{\big[\,(1-z\bar{z})+c_1(1-z\bar{z})^2+O\big((1-z\bar{z})^3\big)\,\big]^2}\\ &=\; |w_{12}|^2\left(\frac{1}{(1-z\bar{z})^2}\;-\;\frac{2c_1}{(1-z\bar{z})}\;+\;O(1)\right)\,.\end{aligned} \tag{A.7}$$

Taking into account of the jacobian, (A.7) gives

$$\frac{1}{F^2}\;=\;\frac{4}{|\xi+i|^4}\left(\frac{|\xi+i|^4}{16(\mathrm{Im}\xi)^2}-\frac{c_1|\xi+i|^2}{4\,\mathrm{Im}\xi}\;+\;O(1)\right)\;=\;\frac{1}{4(\mathrm{Im}\xi)^2}-\frac{c_1}{|\xi+i|^2\,\mathrm{Im}\xi}\;+\;O(1) \tag{A.8}$$

which gives $c_1=0$, when compared to (A.6).

B Details about the Green function

In this appendix we outline some technical details necessary to compute the explicit form of the Green function.

By exploiting the $SU(1,1)$ invariance, we can set the source with charge η/b in the origin and the one with charge ε/b in $t \in (-1,1) \setminus 0$.

By using the definition (3.24) and the expression of $q(z)$ given in (3.18), we can perform explicitly the integrals $I_{ij}(z)$ in the Δ representation of the pseudosphere. They are given by

$$I_{12}(z) \;=\; -\,\frac{tz+z/t-2}{(z-t)\,(z-1/t)} \;+\; C(\eta,t,\beta)\Big(\log(\,z-t\,)-\log(\,z-1/t\,)-\log t^2\Big) \;-\; 2 \tag{B.1}$$

$$\begin{aligned} I_{11}(z) \;=\; z^{2\eta-1}\Bigg\{ &-z\,\frac{2z-t-1/t}{(z-t)\,(z-1/t)} \;-\; \frac{A(\eta,t,\beta)}{2\eta}\,\frac{z}{t}\,F(\,2\eta,1;1+2\eta\,;z/t\,) \\ &+\;\frac{B(\eta,t,\beta)}{2\eta}\,z\,t\,F(\,2\eta,1;1+2\eta\,;z\,t\,)\Bigg\} \end{aligned} \tag{B.2}$$

$$\begin{aligned} I_{22}(z) \;=\; z^{1-2\eta}\Bigg\{ &-z\,\frac{2z-t-1/t}{(z-t)\,(z-1/t)} \;-\; \frac{A(1-\eta,t,\beta)}{2(1-\eta)}\,\frac{z}{t}\,F(\,2-2\eta,1;3-2\eta\,;z/t\,) \\ &+\;\frac{B(1-\eta,t,\beta)}{2(1-\eta)}\,z\,t\,F(\,2-2\eta,1;3-2\eta\,;z\,t\,)\Bigg\} \end{aligned} \tag{B.3}$$

where the functions $A(\eta,t)$, $B(\eta,t)$ and $C(\eta,t)$ are

$$A(\eta,t,\beta) \;=\; 2\,\frac{\eta+(1-\eta)\,t^2}{1-t^2} \;+\; \frac{t\,\beta}{1-t^2} \;=\; B(1-\eta,t,\beta) \tag{B.4}$$

$$C(\eta,t,\beta) \;=\; \frac{1+t^2+\beta\,t}{1-t^2} \;=\; \frac{A(\eta,t,\beta)+A(1-\eta,t,\beta)}{2}\,. \tag{B.5}$$

Notice that only the hypergeometric function of type $F(a,1;a+1\,;x)$ occurs. It is related to a special case of the incomplete beta function [16, 17]

$$\frac{x^a}{a}\,F(a,1;a+1\,;x) \;=\; B_x(a,0) \;=\; \int_0^x \frac{y^{a-1}}{1-y}\,dy\,. \tag{B.6}$$

Now, by inserting the expressions of $I_{ij}(z)$ into (3.43), we find the explicit expression for the propagator

$$
\begin{aligned}
g(z,t) \;=\; & -\frac{1}{2\,(1-2\eta)}\,\frac{1}{(z\bar z)^{\eta}-(z\bar z)^{1-\eta}}\;\times \qquad\qquad \text{(B.7)}\\
& \times\,\Bigg\{ \Big((z\bar z)^{\eta}+(z\bar z)^{1-\eta}\Big)\,\Big[\,C(\eta,t,\beta)\,\Big(\log\omega(z,t)-\log t^2\Big)+4+2\,\mathrm{Re}\,h(t)\,\Big]\\
& \quad + \frac{(z\bar z)^{\eta}}{2\eta}\;\Big[\,A(\eta,t,\beta)\,\Big(z/t\,F(2\eta,1;1+2\eta;z/t)+\text{c.c.}\Big)\\
& \qquad\qquad - B(\eta,t,\beta)\,\Big(z\,t\,F(2\eta,1;1+2\eta;z\,t)+\text{c.c.}\Big)\,\Big]\\
& \quad + \frac{(z\bar z)^{1-\eta}}{2(1-\eta)}\;\Big[\,A(1-\eta,t,\beta)\,\Big(z/t\,F(2-2\eta,1;3-2\eta;z/t)+\text{c.c.}\Big)\\
& \qquad\qquad - B(1-\eta,t,\beta)\,\Big(z\,t\,F(2-2\eta,1;3-2\eta;z\,t)+\text{c.c.}\Big)\,\Big]\Bigg\}
\end{aligned}
$$

where $\omega(z,t)$ is the $SU(1,1)$ invariant

$$
\omega(z,t) \;=\; \left|\,\frac{z-t}{1-z\,t}\,\right|^2 \qquad\qquad t\,\in\,(-1,1) \qquad\qquad \text{(B.8)}
$$

which is related to the geodesic distance on the pseudosphere without sources.
We remark that the expression (B.7) satisfies the equation

$$
-\,\frac{2}{\pi}\,\partial_z\partial_{\bar z}\,g(z,t)\;+\;4\,\mu b^2\,e^{\varphi_{cl}}\,g(z,t)\;=\;\delta^2(z-t) \qquad\qquad \text{(B.9)}
$$

for any β. By employing the expansion [17]

$$
F(\,a,1;a+1;\,w\,)\;=\;a\sum_{k\,=\,0}^{\infty}\frac{(a)_k}{k!}\,\Big(\psi(1+k)-\psi(a+k)-\log(1-w)\Big)\,(1-w)^k \qquad \text{(B.10)}
$$

where $(a)_k\,=\,a(a+1)\ldots(a+k-1)\,=\,\Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol and $\psi(x)=\Gamma'(x)/\Gamma(x)$, one can see that for any β the logarithmic divergence of $g(z,t)$ when $z\to t$ is exactly $-1/2\,\log|z-t|^2$ because the following identity

$$
\frac{\big(1+t^{2\,(1-2\eta)}\big)\,C(\eta,t,\beta)-A(\eta,t,\beta)-A(1-\eta,t,\beta)\,t^{2(1-2\eta)}}{(1-2\eta)\big(1-t^{2\,(1-2\eta)}\big)}\;=\;1\;. \qquad\qquad \text{(B.11)}
$$

To determine β, we impose the monodromy of $g(z,t)$ around $z=t$. When $t<|z|<1$, by using the identity [16]

$$
F(\,a,1;a+1;\,1/w\,)\;=\;\frac{a\pi}{\sin(a\pi)}\,(-\,w)^a+\frac{a}{1-a}\;w\,F(\,1-a,1;2-a;\,w\,) \qquad\qquad \text{(B.12)}
$$

we find that $(-w)^a$ introduces a term that breaks the monodromy. The vanishing of this term leads to the equation

$$A(\eta,t,\beta) - t^{2(1-2\eta)} A(1-\eta,t,\beta) \;=\; 0 \tag{B.13}$$

which allows to get

$$\beta \;=\; -\,2\,\frac{\eta + (\,1-\eta\,)\,t^2 - t^{2\,(1-2\eta)}\,(\,1-\eta+\eta\,t^2\,)}{t\,\big(\,1-t^{2\,(1-2\eta)}\big)} \tag{B.14}$$

which solves the problem of finding the $O(\varepsilon)$ terms of the accessory parameters in the perturbed geometry.

The expression for $\mathrm{Re}\,h(t)$ can be obtained by studying the asymptotic behavior of the Green function $g(z,t)$ when $|z| \to 1$. To get this result, we set $z = e^{i\theta}$ in (B.7) and we analyze its leading term. By using the identity (B.12) and the following one [16]

$$F(\,a,1;a+1;\,w\,) \;=\; \frac{a}{a-1}\,\frac{1}{w}\,\Big(\,F(\,a-1,1;a;\,w\,)-1\,\Big) \tag{B.15}$$

we get

$$F(\,a,1;a+1\,;1/w\,) \;=\; a\,\left(\frac{\pi}{\sin(a\pi)}\,(-w)^a + \frac{w}{1-a} + \frac{w^2}{2-a}\,F(\,2-a,1;3-a\,;w\,)\right) \tag{B.16}$$

which allows us to reduce all the hypergeometric functions occurring in (B.7) to hypergeometric functions with the same parameters but different variables. Then, for any β we find that the leading order in $(1-z\bar{z})$ of $g(z,t)$ contains no contributions from the hypergeometric functions but it includes a term $(-w)^a$ which would break the monodromy. The coefficient of such a term vanishes because of the explicit expression for β found before.

Thus, we have that the leading order in $(1-z\bar{z})$ of the expression contained between the curly brackets in (B.7) is

$$2\,\Big(\,2\,\mathrm{Re}\,h(t) - C(\eta,t,\beta)\log t^2 - 2\,\Big) \tag{B.17}$$

where β is given by (B.14). By requiring the vanishing of (B.17), we find the explicit expression of $\mathrm{Re}\,h(t)$

$$\mathrm{Re}\,h(t) \;=\; \frac{1}{2}\,\left(\frac{1+t^{2\,(1-2\eta)}}{1-t^{2\,(1-2\eta)}}\,\log t^{2\,(1-2\eta)} + 2\right) \tag{B.18}$$

which is also given in (3.44). The vanishing of (B.17) is necessarily imposed by the fact that the divergence of the field φ when $|z| \to 1$ is at most logarithmic.

We will find later that $g(z,t)$ vanishes quadratically when $|z| \to 1$. One could see this

also at this level, but it is much simpler once the expansion of $g(z,t)$ in partial waves is available.

By exploiting the invariance in value of the Green function under rotations, we can easily generalize our formula to the case of complex $t \in \Delta$. Thus, we have that t^2, z/t, $\bar{z}/t$ $z\,t$, and $\bar{z}\,t$ become respectively $t\bar{t}$, z/t, $\bar{z}/\bar{t}$, $z\,\bar{t}$, and $\bar{z}\,t$.

The final expression for the Green function is

$$\begin{aligned} g(z,t) \;\; = \;\; & -\frac{1}{2}\,\frac{1+(z\bar{z})^{1-2\eta}}{1-(z\bar{z})^{1-2\eta}}\,\left\{\,\frac{1+(t\bar{t})^{1-2\eta}}{1-(t\bar{t})^{1-2\eta}}\,\log\omega(z,t)+\frac{2}{1-2\eta}\,\right\} \\ & -\frac{1}{1-(z\bar{z})^{1-2\eta}}\,\frac{1}{1-(t\bar{t})^{1-2\eta}}\,\times \\ \times\,\Bigg\{\,& \frac{(t\bar{t})^{1-2\eta}}{2\eta}\,\frac{z}{t}\,F(\,2\eta,1;1+2\eta;\,z/t\,)+\frac{(z\bar{z})^{1-2\eta}}{2(1-\eta)}\,\frac{z}{t}\,F(\,2-2\eta,1;3-2\eta;\,z/t\,)\;+\;\text{c.c.} \\ -\frac{1}{2\eta}\,& z\,\bar{t}\,F(\,2\eta,1;1+2\eta;\,z\,\bar{t}\,)\;-\;\frac{(z\bar{z})^{1-2\eta}(t\bar{t})^{1-2\eta}}{2(1-\eta)}\,z\,\bar{t}\,F(\,2-2\eta,1;3-2\eta;\,z\,\bar{t}\,)\;+\;\text{c.c.}\,\Bigg\}\;. \end{aligned} \tag{B.19}$$

By using (B.16), this Green function can be written also in the explicit symmetric form given in (3.45). Notice that $g(z,t)$ is regular at $z=0$, as we expect.

It is easy to see that $g(z,t)$ is invariant under $\eta \to 1-\eta$, which is the semiclassical version of the duality $\alpha \to Q-\alpha$. Here it is only a formal invariance because, due to the finite area condition around the sources, $\eta < 1/2$ must hold.

A particular case of the Green function (B.19) is given by the limit $\eta \to 0$, which recovers the geometry of the pseudosphere without curvature singularities. To compute this limit, we use

$$F(\,a,1;a+1;\,w\,)\;=\;1-a\,\log(1-w)+\sum_{k\geqslant 2}(-1)^{k+1}a^k\,\mathrm{Li}_k(w) \tag{B.20}$$

and

$$F(\,2-a,1;3-a;\,w\,)\;=\;-\frac{2}{w^2}\Big(\log(1-w)+w\Big)+\frac{1}{w^2}\sum_{k\geqslant 1}a^k\Big(2\,\mathrm{Li}_{k+1}(w)-\mathrm{Li}_k(w)-w\Big)\;. \tag{B.21}$$

After some algebraic manipulation, we find the following $SU(1,1)$ invariant expression

$$\lim_{\eta\to 0}\,g(z,t)\;=\;-\frac{1}{2}\left(\frac{1+\,\omega}{1-\,\omega}\,\log\,\omega\;+2\right)\;\equiv\;\hat{g}(z,t)\qquad,\qquad\omega\;=\;\left|\,\frac{z-t}{1-z\,\bar{t}}\,\right|^2\;. \tag{B.22}$$

This is the propagator on the regular pseudosphere [6, 1], whose classical background is

$$e^{\varphi_{cl}|_{\eta=0}}\;=\;\frac{1}{\pi\mu b^2(1-z\bar{z})^2}\;. \tag{B.23}$$

To close this appendix, we provide the partial wave expansion of the propagator $g(z,t)$, given in (3.45) or (B.19). For $x \in \mathbb{R}$, we have that

$$\log\left(1 - 2x\cos\theta + x^2\right) \;=\; -2\sum_{m=1}^{\infty} \frac{x^m}{m}\,\cos(m\theta) \qquad x^2 \leqslant 1\,, \qquad x\,\cos\theta \neq 1 \tag{B.24}$$

and

$$\frac{xe^{i\theta}}{a}\,F(\,a,1;a+1;\,xe^{i\theta}\,) \,+\, \text{c.c.} \;=\; 2\sum_{m=1}^{\infty}\frac{x^m}{a-1+m}\,\cos(m\theta) \qquad 0 \leqslant x \leqslant 1\,. \tag{B.25}$$

By using these expansions, we can write the propagator (B.19) as a Fourier series as follows

$$g(z,t) \;=\; \sum_{m=0}^{\infty} g_m(x,y)\,\cos(m\theta) \qquad x \,=\, |z|^2\,, \quad y \,=\, |t|^2 \tag{B.26}$$

where $\theta = \arg(z) - \arg(t)$. The Fourier coefficients can be written in the symmetric and factorized form

$$g_m(x,y) \;=\; \theta(y-x)\,a_m(x)\,b_m(y) + \theta(x-y)\,a_m(y)\,b_m(x) \tag{B.27}$$

where the wave with $m=0$ is given by

$$a_0(x) \;=\; \frac{1+x^{1-2\eta}}{1-x^{1-2\eta}} \qquad b_0(y) \;=\; -\,\frac{1}{2(1-2\eta)}\left(\frac{1+y^{1-2\eta}}{1-y^{1-2\eta}}\,\log y^{1-2\eta} + 2\right) \tag{B.28}$$

while $a_m(x)$ and $b_m(y)$ for $m \geqslant 1$ read

$$a_m(x) \;=\; \frac{x^{m/2}}{1-x^{1-2\eta}}\left(1 - \frac{m-(1-2\eta)}{m+(1-2\eta)}\,x^{1-2\eta}\right) \tag{B.29}$$

$$b_m(y) \;=\; -\,\frac{y^{-m/2}}{m\big(m-(1-2\eta)\big)}\left((1-2\eta)\,\frac{1+y^{1-2\eta}}{1-y^{1-2\eta}}\,(1-y^m) - m(1+y^m)\right)\,. \tag{B.30}$$

This expansion allows to find the asymptotic behavior of the Green function (B.19) at infinity in a simple way. Indeed, since for any m

$$b_m(y) \;=\; O\big((1-y)^2\big) \qquad \text{when} \qquad y \to 1 \tag{B.31}$$

then also the propagator vanishes quadratically at infinity

$$g(z,t\,) \;=\; O\big((1-z\bar{z})^2\big) \qquad \text{when} \qquad |z| \to 1\,. \tag{B.32}$$

From the behavior

$$a_m(y) \,\propto\, \frac{1}{1-y} \qquad \text{when} \qquad y \to 1 \tag{B.33}$$

we see that $g(z,t)$ given in (B.19) is the unique Green function which does not diverge at infinity.
The Fourier expansion simplifies also the analysis of the limit $\eta \to 0$. Indeed, taking the expressions of $a_m(x)$ and $b_m(y)$ in this limit, with a special care for the case $m=1$, one can easily verify that they reproduce the Fourier expansion of the propagator (B.22), which was found in [8] and was used there to perform a three loop calculation on the background of the regular pseudosphere.

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